Interacting Boson Gases and Large Deviation Principles

by

Matthew Dickson

Thesis

Submitted for the degree of

Doctor of Philosophy

Mathematics Institute

The University of Warwick

September 2019

Contents

Acknowledgments

Firstly I owe deep thanks to my supervisor Stefan Adams for his patience and valuable guidance over the last few years. Beyond the academic, our tangential conversations have helped me keep perspective. Roger Tribe and Steffen Dereich have my gratitude for their perseverance in reading and examining my thesis and highlighting many of my oversights.

I would like to thank the Engineering and Physical Sciences Research Council (EP-SRC) for funding the MASDOC DTC at the University of Warwick via grant No. EP/HO23364/1.

Additional thanks is due to my fellow members of my MASDOC cohort, in particular those resident in B0.24, for those aptly timed coffee breaks and distractions from the horrors of missing minus signs.

Lastly, and most importantly, I thank my parents and Marianna for their lasting support and confidence.

Declarations

The work in this thesis was conducted by the author during the period October 2016 - September 2019 at the University of Warwick, in collaboration with Dr Stefan Adams. Elements of Chapters 4 and 5 have previously appeared in [AD18], and Section 6.2 in [AD19]. Where we make use of work not our own, or rework established arguments, we write (for instance): "we use the techniques of [ACK11]". To the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise stated. This thesis has not been submitted for a degree at any other university.

Abstract

We consider probabilistic models for interacting bosons at a positive temperature in the thermodynamic limit with random particle density. The Loop Soup representation is a marked point process arising from the Feynman-Kac formula for the grand canonical partition function for a Bose gas, and the Cycle Count representation is in turn found by projecting onto the space of mark-type densities. Interacting versions of the Loop Soup and Cycle Count models are found by applying Varadhan-like arguments. Large deviation principles are found for a variety of interactions, which provide representation formulae for the thermodynamic limit of the pressure and a means to study Bose-Einstein condensation in these models.

Notation

Mathematics

$\mathbb{N}$	Positive natural numbers, $\{1,2,3,\ldots\}$
$\mathbb{N}_0$	Non-negative natural numbers, $\{0,1,2,\ldots\}$
$\mathfrak{S}_N$	Set of permutations of $\{1,\ldots,N\}$
$\mathbb{R}$	Real numbers
$\mathbb{R}_+$	Non-negative real numbers
$\varnothing$	Empty set
$\#$	Cardinality of a discrete set
$\mathrm{int}(A)$	Interior of the set A
$\mathrm{cl}(A)$	Closure of the set A
$\mathbb{1}\{A\}$	Indicator function of the event A
$(x)_\pm$	Positive/negative component of the real number x
$\lfloor x \rfloor$	Integer part of x
$n!$	n factorial
$\delta_{i,j}$	Kronecker delta function
$\mathtt{Tr}[A]$	Trace of the operator A
$\langle s,x\rangle$	Inner product of s and x

Asymptotics

$a_n = o(b_n)$	The ratio $\frac{a_n}{b_n} \to 0$ as $n \to \infty$
$a_n = O(b_n)$	The ratio $\frac{a_n}{b_n}$ is bounded as $n \to \infty$
$a_n \gg b_n$	The ratio $\frac{a_n}{b_n} \to +\infty$ as $n \to \infty$
$a_n \sim b_n$	If $a_n = b_n(1+o(1))$

Probability

$\mathbb{E}[f]$	Expectation
$\mathtt{Var}[f]$	Variance
$\boldsymbol{\mu}_{x,y}^{(t,\mathrm{bc})}$	Non-normalised Brownian bridge measure
$\mathcal{L}$	Logarithmic moment generating function
$\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$	Radon-Nikodym derivative

Statistical Mechanics

β	Inverse temperature
α, μ	Chemical potential
$Z_\Lambda(\beta, N)$	Canonical partition function
$Z_\Lambda(\beta, \mu)$	Grand-canonical partition function

Loop Ensemble

E	Mark space
Ω	Counting measures on $\mathbb{R}^d \times E$
$\mathcal{P}_\theta$	Shift-invariant probability measures on Ω

Chapter 1

Introduction

In broad and vague terms, statistical mechanics is the study of a large (say 10^{23}) number of objects with some prescribed microscopic (small scale) behaviour, with the aim of succinctly describing some macroscopic (large scale) behaviour. In the early history of kinetic and atomic theory, figures such as Daniel Bernoulli, Herapath, Joule, Krönig, and Clausius had shown that the relations between pressure, temperature and density in a non-interacting gas could be explained by supposing particles moved with uniform velocity in straight lines. The first statistical law of physics was given by James Clerk Maxwell in [Max60], in which he considered an equilibrium gas of small hard-core particles with perfectly elastic collisions and gave the proportion of particles having a certain velocity in a certain range. This was later generalised by Ludwig Boltzmann in [Bol72], and Maxwell-Boltzmann statistics describes the distribution of non-interacting particles in an equilibrium gas over energy states.

One of the most interesting phenomenon in statistical mechanics is the phase transition. This is where for one range of parameters we see one type of macroscopic behaviour, whilst for another range another type emerges. Perhaps the most well-known model in statistical mechanics is the Ising model. Introduced by Wilhelm Lenz in [Len20], this is a model of a ferromagnetic lattice where the polarity of each site is encouraged to align with its nearest neighbours. In the presence of a strong external field, naturally all the sites align and the system is considered *magnetised*. In particular, there is a long range correlation between the polarity of sites. Intuition may suggest that as the strength of this external field approaches zero, then this long range correlation should also vanish. In fact, Lenz's student Ernst Ising showed in [Isi25] that this is what happens for the 1-dimensional lattice. However, Rudolf Peierls in [Pei36] and Lars Onsager in [Ons44] showed that this intuition is incorrect for the 2-dimensional lattice Ising model. They proved that for systems below a critical temperature, the long range correlations persist even as the external field vanishes. This difference between the system above this critical temperature (where residual magnetisation vanishes) and below this critical temperature is the archetype of a phase transition. The Ising model remains one of the most studied models of statistical physics.

The phase transition which we shall be concerned with in this thesis is *Bose-Einstein condensation* (BEC). Satyendra Nath Bose described the statistical distribution of the modes of photons and other so-called bosons in [Bos24], and after correspondence with Bose, Albert Einstein studied the non-interacting (or *ideal*) boson gas. In [Ein24], he observed that at moderate temperatures any given energy state only contained a microscopic proportion of the particles, but that there was a critical temperature below which a macroscopic proportion (a positive fraction) of the total particle number occupied precisely the lowest single-particle energy state. Einstein remarked: "A separation is effected; one part condenses, the rest remains a saturated ideal gas."[1] Whilst the Ising model phase transition had been expected - the Curie point was an established physical phenomenon - prior to Einstein's observation no one had expected such a condensation and his result initially attracted little enthusiasm. Beyond the difficulties of any experiment that would produce such a condensate, it is not clear how robust the result would be to the introduction of an interaction. The attention given to BEC grew after Fritz London's observation in [Lon38] of a similar transition in liquid helium, and Oliver Penrose and Lars Onsager gave a concrete definition of BEC for interacting gases in [PO56]. Although work on BEC continued throughout the latter half of the twentieth century, it has experienced a resurgence since 1995. With the development of new cooling technology, two independent teams demonstrated the existence of the Bose-Einstein

[1] This English translation of an Einstein quote is due to [Pai82].

condensate. The group of Eric Cornell and Carl Wieman condensed a vapour of rubidium-87 [And+95], and soon after the team of Wolfgang Ketterle condensed a gas of sodium-23 [Dav+95]. These achievements led to Cornell, Wieman and Ketterle sharing the 2001 Nobel Prize for Physics [Nob01] "for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates."

For many studies of boson gases, the model is formulated with the energy spectrum foremost in the mind. We shall be taking an alternative perspective. In [Fey53], Richard P. Feynman used the Feynman-Kac formulae to describe the partition function of a boson gas as an expectation over a random ensemble of loops. He also conjectured that BEC corresponds to the emergence of 'long loops' in this ensemble, and that the condensate density is given by the number of particles residing in these long loops. This phenomenon is also signalled by a loss of probability mass in the distribution of the 'finite' cycles. See [Süt02] for proofs of this coincidence in the ideal Bose gas and some mean-field models. A closely related line of research is studying the effect of the symmetrisation in random permutation and random partition models, see [Ver96], [BCMP05], [AD08; AK08; Ada08], or in spatial random permutation models going back to [Fic91].

In this thesis we shall explore this random loop ensemble, in particular investigating its behaviour under various interactions. Forming an integral part of our analysis will be the techniques of large deviations. In deriving the large deviation principles (LDPs) and the associated rate functions, we will also derive the thermodynamic pressure, and in finding the minimisers of our rate functions we will often find we get convergence-in-probability results for free.

1.1 The Quantum Mechanical Bose Gas

Here we describe the usual quantum model for the Bose gas, before we introduce the Feynman-Kac formula and our loop ensemble. Let us, for a moment, recall some classical mechanics. When we consider the *canonical ensemble*, we have N particles in a region $\Lambda \subset \mathbb{R}^d$ with volume $|\Lambda| < \infty$. Then the configuration space is given by

$\Gamma_{\Lambda,N} = \left(\Lambda \times \mathbb{R}^d\right)^N$, and a configuration is given by (x,p) where $x \in \Lambda^N$ corresponds to the particle positions, and $p \in \mathbb{R}^{dN}$ corresponds to their momenta. To get the energy of a configuration, we consider the Hamiltonian

$$H_N\left(x,p\right) := \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{1\leq i<j\leq N} V\left(\left|x_i - x_j\right|\right), \qquad (x,p) \in \Gamma_{\Lambda,N}. \tag{1.1.1}$$

Here m is the mass of a particle and $V : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ is a pair-potential that describes particle interactions. If we set the *inverse temperature* β, then the probability assigned to a Lebesgue-measurable subset $E \subset \Gamma_{\Lambda,N}$ is then given by the Boltzmann weights as so:

$$\mathbb{P}_{\beta,\Lambda,N}\left(E\right) = \frac{1}{N! Z_\Lambda\left(\beta, N\right)} \int_E \exp\left(-\beta H_N\left(x,p\right)\right) \mathrm{d}x\mathrm{d}p, \tag{1.1.2}$$

where $Z_\Lambda\left(\beta, N\right)$, the *canonical partition function*, normalises the measure. Note that the $N!$ factor has no influence on $\mathbb{P}_{\beta,\Lambda,N}$, but it gives the correct Boltzmann counting to resolve the Gibbs paradox. This is discussed further in [Ada06], but is related to the indistinguishability of identical particles and is better justified in quantum mechanics.

Since we wish to consider at least $N \sim 10^{23}$, we may as well "go the whole hog" and take $N \to \infty$. To do so we introduce the *thermodynamic limit*. For the canonical ensemble this means we consider two sequences Λ_n, N_n such that $N_n/|\Lambda_n| \to \varrho$ and consider the limiting measure. In particular, we could be interested in the thermodynamic function

$$f\left(\beta, \varrho\right) := \lim_{n\to\infty} \frac{-1}{\beta|\Lambda_n|} \log Z_{\Lambda_n}\left(\beta, N_n\right), \tag{1.1.3}$$

called the *free energy*.

An alternative to the canonical ensemble is the *grand canonical ensemble*. Now instead of fixing the total particle number, we allow it to fluctuate around some mean particle number. Our configuration space is then $\Gamma_\Lambda = \bigcup_{N=0}^{\infty} \Gamma_{\Lambda,N}$, and we assign an extra weighting $\exp\left(\beta\mu N\right)$ to configurations in $\Gamma_{\Lambda,N}$. Physically, the grand canonical ensemble can be imagined as describing a region Λ with permeable walls in a much larger bath. The parameter $\mu \in \mathbb{R}$, called the *chemical potential*, is then an energetic encouragement or discouragement for particles to enter the region from the bath or vice versa. Note that we are not guaranteed a finite measure for all values of μ. The normalisation of this new measure is called the *grand canonical partition function*, and is given by

$$Z_\Lambda(\beta, \mu) = \sum_{N=0}^{\infty} \mathrm{e}^{\beta \mu N} Z_\Lambda(\beta, N). \tag{1.1.4}$$

Note that we can use this partition function to directly find out some properties of the system. If we let $\mathcal{N}$ be the random variable giving the total particle number, then under the grand canonical probability measure we have

$$\mathbb{E}_{\beta,\Lambda,\mu}[\mathcal{N}] = \frac{1}{\beta}\frac{\partial}{\partial \mu} \log Z_\Lambda(\beta, \mu), \qquad \mathrm{Var}_{\beta,\Lambda,\mu}[\mathcal{N}] = \frac{1}{\beta^2}\frac{\partial^2}{\partial \mu^2} \log Z_\Lambda(\beta, \mu). \tag{1.1.5}$$

The thermodynamic limit now consists of simply fixing μ and having $|\Lambda_n| \to \infty$. The main thermodynamic function is then

$$p(\beta, \mu) := \lim_{n\to\infty} \frac{1}{\beta|\Lambda_n|} \log Z_{\Lambda_n}(\beta, \mu), \tag{1.1.6}$$

called the *pressure*.

Even popular quantum physics can tell us that something is going to go wrong with these constructions if we try to apply them to quantum systems. The Heisenberg uncertainty principle dictates that it is impossible to simultaneously observe the position and momentum of a particle, and therefore it no longer makes sense to consider the configurations spaces $\Gamma_{\Lambda,N}$. Instead, our state is described by a wave-function $\psi(x)$. In the canonical ensemble, this is a complex valued function in the Hilbert space $L^2\left(\Lambda^N\right)$, and the probability that the system is in the measurable set $A \subset \Lambda^N$ is proportional to $\int_A |\psi(x)|^2 \mathrm{d}x$. Now the Hamiltonian of the system is given by a *Schrödinger operator* acting on $L^2\left(\Lambda^N\right)$:

$$H_N \psi = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i^{(\mathrm{bc})} \psi + \sum_{1\le i<j\le N} V(|x_i - x_j|)\, \psi, \tag{1.1.7}$$

where $\hbar \approx 1.05 \times 10^{-34} \mathrm{kg\,m^2 s^{-1}}$ is the reduced Planck constant, and $\Delta_i^{(\mathrm{bc})}$ is the Laplacian associated with particle i under some boundary conditions 'bc'. Permissible wave functions are the eigenfunctions of the Hamiltonian, and their energies are the corresponding eigenvalues. The Hamiltonian in (1.1.7) is analogous to the classical version (1.1.1), where the momenta p_i have been replaced with momentum operators $-i\hbar\nabla_i$.

We now introduce the role *bosons* and *fermions* play in our analysis. Suppose we have a transposition operator T, that swaps the labels of the i and j particles (with $i \neq j$). Clearly $T^2 = I$, the identity operator, and therefore the eigenvalues of T are $+1$ and -1. The eigenfunctions with eigenvalue $+1$ are the symmetric wavefunctions and describe gases of bosons, whereas the eigenfunctions with eigenvalue -1 are the anti-symmetric wavefunctions and describe gases of fermions. The anti-symmetry of the fermionic wavefunction leads to many interesting and important physical phenomenon - including the Pauli exclusion principle - but we shall be concerned with bosons. By focussing solely on bosons, we restrict our Hilbert space $L^2\left(\Lambda^N\right)$ to the sub-Hilbert space $L^2_+\left(\Lambda^N\right) \subset L^2\left(\Lambda^N\right)$ of symmetric L^2 functions.

Since the eigenvalues of H_N describe the permissible energy levels of the system, the natural analogue of the classical canonical partition function is the following *quantum canonical partition function*

$$Z_\Lambda(\beta, N) = \mathrm{Tr}_{L^2_+(\Lambda^N)}\left[\exp\left(-\beta H_N\right)\right], \tag{1.1.8}$$

where Tr denotes the trace of an operator.

For the grand canonical ensemble of bosons, the quantum state space is now the Fock space of symmetric wave functions, and the Hamiltonian operator is given by its action on each N-particle subspace:

$$\mathcal{F}_+(\Lambda) = \bigoplus_{N=0}^{\infty} L^2_+\left(\Lambda^N\right), \qquad \mathcal{H} = \bigoplus_{N=0}^{\infty} H_N. \tag{1.1.9}$$

If we let $\mathcal{N}$ be the particle number operator ($\mathcal{N}\psi = N\psi$ if $\psi \in L^2\left(\Lambda^N\right)$), then the *quantum grand canonical partition function* is given by

$$Z_\Lambda(\beta,\mu) = \mathrm{Tr}_{\mathcal{F}_+(\Lambda)}\left[\exp\left(\beta\mu\mathcal{N} - \beta\mathcal{H}\right)\right] \tag{1.1.10}$$

$$= \sum_{N=0}^{\infty} \mathrm{e}^{\beta\mu N} Z_\Lambda(\beta,N). \tag{1.1.11}$$

Like for the classical versions, an analysis of the Bose gas can now proceed by analysing thermodynamic functions defined using the canonical and grand canonical partition function, see for example [ZUK77].

1.1.1 Bose-Einstein Condensation

In [Ein24], Einstein studied the *ideal Bose gas*, that is the Bose gas with interaction $V \equiv 0$. In this case the grand canonical partition function factorises into contributions from the single particle energy levels $\epsilon_0 \leq \epsilon_1 \leq \ldots$. In the case of $\Lambda = [-L/2, L/2]^d \subset \mathbb{R}^d$ with periodic boundary conditions, these energy levels are proportional to $\underline{n}^2/L^2$ with $\underline{n} \in \mathbb{Z}^d$. The partition function is

$$Z_\Lambda(\beta,\mu) = \prod_{i\geq 0} \frac{1}{1-\exp\left(-\beta\left(\epsilon_i - \mu\right)\right)}. \tag{1.1.12}$$

Note that this is finite only if $\mu < \epsilon_0$, and in the thermodynamic limit $\epsilon_0 \to 0$. The expected particle number is then given by

$$\mathbb{E}_{\beta,\Lambda,\mu}[N] = \sum_{i\geq 0} \frac{1}{\exp\left(\beta\left(\epsilon_i-\mu\right)\right)-1}. \tag{1.1.13}$$

In taking the thermodynamic limit, we then approximate the sum with an integral, and we have for $\mu < 0$

$$\varrho(\beta,\mu) = \lim_{L\to\infty} \frac{\mathbb{E}_{\beta,\Lambda,\mu}[N]}{L^d} = \frac{1}{(2\pi\hbar)^d}\int \mathrm{d}\mathbf{p} \frac{1}{\exp\left(\beta\left(\frac{\mathbf{p}^2}{2m}-\mu\right)\right)-1}. \tag{1.1.14}$$

However, note that for all $\mu < 0$ we have $\varrho(\beta,\mu) < \left({}^{m}/{}_{2\pi\beta\hbar}\right)^{d/2} \zeta\left({}^{d}/{}_{2}\right) =: \varrho_c(\beta)$, where $\zeta(s) = \sum_{j\geq 1} j^{-s}$ is the Riemann-zeta function. Note that for $d \geq 3$, $\varrho_c < \infty$ and we have an upper bound on the particle density. This is absurd. The resolution of this paradox is that the excess particles all go into the lowest energy state, and mathematically this means that we have to let μ vary with Λ. In particular, if $\varrho > \varrho_c$ we choose $\mu = \mu_\Lambda$ such that $\mathbb{E}_{\beta,\Lambda,\mu_\Lambda}[N] = |\Lambda|\varrho$. Then we find

$$\varrho = \varrho_c(\beta) + \varrho_0, \qquad \varrho_0 = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \frac{1}{\exp(\beta(\epsilon_0 - \mu_\Lambda)) - 1}, \tag{1.1.15}$$

where ϱ_0 is the density of the "condensate." If $\varrho_0 > 0$, then we say that the single particle ground state is macroscopically occupied. As [LSSY05] highlights, this condensation is interesting because it happens at positive temperature. For any statistical mechanics model at zero temperature (formally $\beta = \infty$) we would expect all the particles to reside in the ground state. In the thermodynamic limit the energy levels become closer and closer together, and if we have positive temperature we may expect particles to somehow be evenly spread around an ever larger number of energy states in some energy-neighbourhood of the ground state. This does not happen here. We can choose ϱ and β such that for each and every excited energy level we do not have macroscopic occupation, i.e.

$$\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \frac{1}{\exp(\beta(\epsilon_i - \mu_\Lambda)) - 1} = 0, \tag{1.1.16}$$

and yet ϱ_0 is strictly positive.

The above analysis of the ideal Bose gas does not generalise nicely to the interacting version because it is not obvious what is now meant by a macroscopic occupation of a single-particle state - the eigenfunctions of the Hamiltonian H_N are no longer products of single particle states. The concept of BEC was first generalised to interacting models in [PO56] by Penrose and Onsager with their study of the one-particle reduced density matrix. Following [LSSY05], we introduce the short-hand notation $\mathbf{X} = (\mathbf{x}_2, \ldots, \mathbf{x}_N)$ and $\mathrm{d}\mathbf{X} = \prod_{j=2}^{N} \mathrm{d}\mathbf{x}_j$ and consider some normalised wave function Ψ. Then the one-particle reduced density matrix is given by

$$\gamma_N(\mathbf{x}, \mathbf{x}') = N \int \Psi(\mathbf{x}, \mathbf{X}) \Psi(\mathbf{x}', \mathbf{X}) \, \mathrm{d}\mathbf{X}, \tag{1.1.17}$$

and $\int \gamma_N(\mathbf{x}, \mathbf{x}) \, \mathrm{d}\mathbf{x} = \mathrm{Tr}[\gamma_N] = N$. A system is then said to have BEC in the ground state if the operator γ_N corresponding to the ground state wave function of H_N has an eigenvalue of order N in the thermodynamic limit. In [PS08, pp.396-7], it has been shown that this is equivalent to having

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \lim_{|\Lambda|\to\infty} \gamma_N\left(\mathbf{x},\mathbf{x}'\right) \neq 0, \tag{1.1.18}$$

and furthermore that in the case of the ideal gas,

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \lim_{|\Lambda|\to\infty} \gamma_N\left(\mathbf{x},\mathbf{x}'\right) = \varrho_0. \tag{1.1.19}$$

Whilst this generalisation is appealing, rigorously proving this behaviour remains a formidable task. Lieb and Seiringer in [LS02] provide the only rigorous proof of BEC in the continuum for a class of trap potentials with two-body repulsive interactions in the dilute limit, and rigorous proofs for many-body Hamiltonians with genuine interactions have been restricted to hard-core bosons on a lattice at half-filling (see [DLS78] and [KLS88]).

1.2 Feynman-Kac Formulae and Loop Ensembles

Feynman introduced the Feynman-Kac formulae in [Fey48] and [Fey53] in order to make his abstract path integrals rigorous, and in the latter he derived a formula for the partition function of the Bose gas as an integral over an ensemble of trajectories, distributed as interacting Brownian bridges.

Let us begin with the canonical ensemble. Now given an operator $\exp\left(-tH\right)$, we want to find a kernel $K_t\left(x,y\right)$ such that

$$\left(\mathrm{e}^{-tH}f\right)(x) = \int_{\Lambda_N} K_t\left(x,y\right) f\left(y\right) \mathrm{d}y, \qquad f\in L^2\left(\Lambda^N\right). \tag{1.2.1}$$

Decompose the Hamiltonian into a kinetic part and and an interaction part: $H = H_0 + V$. Now for the ideal gas $H = H_0 = -\frac{1}{2}\Delta$, and in the infinite volume limit it is well known that our desired kernel is given by the heat kernel

$$K_t\left(x,y\right)=\frac{1}{\left(2\pi t\right)^{\frac{d}{2}}}\exp\left(-\frac{1}{2t}|x-y|^2\right). \tag{1.2.2}$$

The connection to stochastic processes becomes apparent upon the realisation that g_t is the transition kernel of a d-dimensional Brownian motion. Results of this type can be extended to interacting models with $V\not\equiv 0$. One such is the following proposition.

Proposition 1.2.1. *Let $H=-\frac{1}{2}\Delta+V$ with $V\in\mathcal{C}_0^\infty\left(\mathbb{R}^d\right)$. Then*

$$\left(\mathrm{e}^{-tH}f\right)(x)=\mathbb{E}_x\left[\exp\left(-\int_0^t V\left(B_s\right)\mathrm{d}s\right)f\left(B_t\right)\right],\qquad f\in L^2\left(\mathbb{R}^d\right), \tag{1.2.3}$$

where $\mathbb{E}_x$ is the expectation with respect to the Wiener measure $\mathbb{P}_x$ of a Brownian motion started at $x\in\mathbb{R}^d$, $B_0=x$.

This is Theorem 3.30 from [LHB11], which contains a proof as well as an in-depth treatment of Feynman–Kac formulae under weaker assumptions.

A little more work is required to find the canonical partition function. We need to use (1.2.3) to find the trace of e^{-tH}, and it is here that the bosonic behaviour manifests itself. Let us denote by $\mathfrak{S}_N$ the set of permutations of $\{1,\dots,N\}$, and write $\boldsymbol{\mu}_{x,y}^{(t,\mathrm{bc})}(\cdot)=\mathbb{P}_x^{(\mathrm{bc})}\left[\cdot,B_t=y\right]$ for the non-normalised Brownian bridge measure from x to y over time horizon $t>0$ obeying boundary condition bc. In this thesis we will consider three different boundary conditions: $\mathrm{bc}\in\{\varnothing,\mathrm{Dir},\mathrm{per}\}$. These "empty," "Dirichlet," and "periodic" boundary conditions will be described in more detail in Section 2.2, and in Appendix A. For $\mathrm{bc}=\varnothing$, we let the bridge leave Λ and move as normal, for $\mathrm{bc}=\mathrm{Dir}$ we restrict the bridges to remain within Λ, and for $\mathrm{bc}=\mathrm{per}$ we treat Λ as a torus.

Theorem 1.2.2. *Let $H=-\frac{1}{2}\Delta^{(\mathrm{bc})}+V$ be the Hamiltonian of a Bose gas where V*

decays sufficiently fast. The the canonical partition function has the representation

$$Z_\Lambda^{(\mathrm{bc})}(\beta, N) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \int_\Lambda \mathrm{d}x_1 \cdots \int_\Lambda \mathrm{d}x_N \\ \bigotimes_{i=1}^{N} \boldsymbol{\mu}_{x_i, x_{\sigma(i)}}^{(\beta,\mathrm{bc})} \left[\exp\left\{ - \sum_{1 \le i < j \le N} \int_0^\beta V(|B_s^{(i)} - B_s^{(j)}|)\, \mathrm{d}s \right\} \right]. \quad (1.2.4)$$

See [Fey53] for the classical reference, or [Gin71] for a rigorous account. It is now a simple matter to calculate the grand canonical partition function by combining Theorem 1.2.2 with our earlier equality (1.1.11).

Now since any permutation decomposes into cycles, and due to the Markov property, the N Brownian bridges in (1.2.4) concatenate into loops of various time horizons in $\beta\mathbb{N}$. These loops have an anchor that serves as the start and end of the loop, and which is uniformly distributed over Λ. It will prove useful to define a "anchor number" and a "particle number" for any configuration of loops. The anchor number is naturally the number of anchors in a given region, whilst the particle number is given by the number of the original Brownian bridges that were used to construct the loops associated with these anchors. The necessity of this distinction is highlighted when we consider the interaction energy of a configuration. The interaction energy is given by imagining a particle moving at parameter speed along each of the original bridges. For example, a loop constructed from three legs has three particles moving around it, each a parameter distance β from each other. The energy of the configuration is then the integral of the energy of these particles under potential V over the parameter time interval $[0, \beta]$. Now the canonical partition function is equal to that of an interacting marked Poisson point process with the condition that the particle number is fixed - the anchors are the points and the loops are the marks in this description.

To describe the grand canonical ensemble we then remove the condition on the particle number, but add a chemical potential energy term that depends linearly upon this particle number. The effect of this is to change the probability that an anchor's loop has a given time horizon. As seen in [AD18] this allows us to give the grand canonical partition function for a Bose gas with pair-wise interactions solely in terms of an interacting marked Poisson point process (with no extra condition). We present this formulation in more detail in Section 2.2.

1.3 Summary of Contents and Structure

In this thesis the main objective will be to describe - in various ways - the interacting loop ensemble in the grand canonical ensemble. Our main tool in doing this will be the *large deviation principle*. This will allow us to prove the thermodynamic pressure, derive a condensate density, and arrive at other convergence results for a variety of interactions.

Chapter 2 begins with an overview of the large deviation techniques that we will later use. These will allow us to build LDPs from scratch, as well as derive LDPs for measures constructed by tilting a previously understood measure. Section 2.2 then presents the construction for the full random loop ensemble as an interacting marked Poisson point process and introduces the *stationary empirical measure*. Naturally, this is a very high-dimensional object, and so we will spend much of our time discussing a slightly simpler model. In Section 2.3 we project the full process down to one on ℓ_1 where we only keep track of the number of anchors with each length of loop (or cycle). Whilst this loses much of the detail of the original model, we give a number of interaction energies that are expressible in terms of the *empirical cycle count*. In particular, we consider an interaction analogous to the Huang-Yang-Luttinger (HYL) interaction derived in [HYL57] as an approximation of hard-core interactions.

The thesis then proceeds by analysing models with decreasing detail of resolution, but increasing solvability. First, we analyse the full interacting loop ensemble in Chapter 3. Here we take the arguments of [ACK11] for the canonical ensemble and apply it to the grand canonical ensemble. Whilst we are successful in deriving a LDP (and proving the existence of a thermodynamic pressure), the resulting variation expression is difficult to analyse. Nevertheless, we do achieve progress in deriving an equivalence between the grand-canonical and canonical ensembles in Section 3.2. Chapters 4 and 5 are expansions upon previous work in [AD18]. In the former we derive LDPs for the empirical cycle count under a variety of interactions. This also gives us variational expressions, but because these are now only over ℓ_1, they are much more tangible than for the full model. In Chapter 5 we perform detailed analysis of these LDPs and pressure representations. In particular, we are able to study the BEC phenomena in these models and derive explicit expressions for the condensate density. One last decrease in resolution is taken in Chapter 6. Some of the work in this chapter is related to the author's work appearing in [AD19]. Here we take inspiration from [BLP88] and consider the total particle number supported on loops with length below some cut-off and the total particle number supported on loops with length above that cut-off. This is not a simple application of the Contraction Principle (Lemma 2.1.15) because the total particle density is not a continuous transformation of the empirical cycle count. The focus in this chapter will be on a partial-HYL interaction that we show preserves many of the features of the full HYL interaction considered in the previous chapters whilst being easier to analyse. We also use similar techniques to generalise the analysis of the true momentum pace HYL model by [BLP88].

Whilst this thesis aims to be self-contained and to be read in a linear manner, we defer some technical or tangential details and analysis of different boundary conditions to the appendices.

Chapter 2

Definitions and Preliminaries

2.1 Large Deviation Techniques

In this thesis, we take advantage of large deviation techniques to prove limits such as the thermodynamic pressure, and central limit theorem type results. Here we present some of the results that will play central roles in our proofs.

Definition 2.1.1. Let $\mathcal{X}$ be a Polish space (a separable completely metrizable topological space). The function $\mathcal{I} : \mathcal{X} \to [0, \infty]$ is called a *rate function* if $\mathcal{I} \not\equiv \infty$, $\mathcal{I}$ is lower semicontinuous, and $\mathcal{I}$ has compact level sets. A sequence of probability measures $\{P_N\}_{N\in\mathbb{N}}$ on $\mathcal{X}$ is said to satisfy a *large deviation principle* with rate r_n and rate function $\mathcal{I}$ if

$$\limsup_{N\to\infty} \frac{1}{r_N} \log P_N(C) \leq - \inf_{x\in C} \mathcal{I}(x), \qquad \forall C \subset \mathcal{X} \text{ closed}, \tag{2.1.1}$$

$$\liminf_{N\to\infty} \frac{1}{r_N} \log P_N(O) \geq - \inf_{x\in O} \mathcal{I}(x), \qquad \forall O \subset \mathcal{X} \text{ open}. \tag{2.1.2}$$

Definition 2.1.2. Let (X, τ_X) be a general topological space and Y be a totally ordered space. Then $f : X \to Y$ is said to be *lower semicontinuous* with respect to τ_X if the preimage $f^{-1}(\{y : y > c\}) \in \tau_X$ for all $c \in Y$. If X is also a metric space, then

$$\liminf_{x'\to x} f(x') \geq f(x) \qquad \forall x \in X. \tag{2.1.3}$$

Also, $f : X \to Y$ is said to be *upper semicontinuous* with respect to τ_X if the preimage $f^{-1}(\{y : y < c\}) \in \tau_X$ for all $c \in Y$. If X is also a metric space, then

$$\limsup_{x' \to x} f(x') \leq f(x) \qquad \forall x \in X. \tag{2.1.4}$$

Lemma 2.1.3 (Varadhan's Lemma Lower Bound). *Let $\{\mathbb{P}_N\}_{N\in\mathbb{N}}$ be a sequence of probability measures satisfying a large deviation lower bound on $\mathcal{X}$ with rate r_N and rate function $\mathcal{I}$. If $F : \mathcal{X} \to \mathbb{R}$ is upper semicontinuous, then*

$$\liminf_{N\to\infty} \frac{1}{r_N} \log \mathbb{E}_N \left[\mathrm{e}^{-r_N F}\right] \geq -\inf_{x\in\mathcal{X}} \{I(x) + F(x)\}. \tag{2.1.5}$$

Lemma 2.1.4 (Varadhan's Lemma Upper Bound). *Let $\{\mathbb{P}_N\}_{N\in\mathbb{N}}$ be a sequence of probability measures satisfying a large deviation upper bound on $\mathcal{X}$ with rate r_N and rate function $\mathcal{I}$. If $F : \mathcal{X} \to \mathbb{R}$ is lower semicontinuous and obeys the tail condition*

$$\lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{r_N} \mathbb{E}_N \left[\mathrm{e}^{-r_N F} \mathbb{1}\{F \leq M\}\right] = -\infty, \tag{2.1.6}$$

then

$$\limsup_{N\to\infty} \frac{1}{r_N} \log \mathbb{E}_N \left[\mathrm{e}^{-r_N F}\right] \leq -\inf_{x\in\mathcal{X}} \{I(x) + F(x)\}. \tag{2.1.7}$$

Proof of Lemmas 2.1.3 and 2.1.4. These follow from [DZ09, Lemmas 4.3.4 & 4.3.6].

□

Theorem 2.1.5 (Varadhan's Lemma). *Let $\{\mathbb{P}_N\}_{N\in\mathbb{N}}$ be a sequence of probability measures satisfying a large deviation principle on $\mathcal{X}$ with rate r_N and rate function $\mathcal{I}$. Now if F is continuous and obeys the tail condition* (2.1.6)*, then*

$$\lim_{N\to\infty} \frac{1}{r_N} \log \mathbb{E}_N \left[\mathrm{e}^{-r_N F}\right] = -\inf_{x\in\mathcal{X}} \{I(x) + F(x)\}, \tag{2.1.8}$$

and the sequence of measures $\left\{P_N^F\right\}_{N\in\mathbb{N}}$ defined by

$$\mathbb{P}_N^F(S) = \frac{\int_S \mathrm{e}^{-r_N F(x)} \mathbb{P}_N(\mathrm{d}x)}{\mathbb{E}_N\left[\mathrm{e}^{-r_N F}\right]}, \quad S \subset \mathcal{X} \text{ Borel}, \tag{2.1.9}$$

satisfies the large deviation principle on $\mathcal{X}$ with rate r_N and rate function

$$\mathcal{I}^F(x) = \mathcal{I}(x) + F(x) - \inf_{\mathcal{X}} \{\mathcal{I} + F\}. \tag{2.1.10}$$

PROOF. The large deviation principle follows easily once one has proven the lower and upper bounds of Lemmas 2.1.3 and 2.1.4. The behaviour of the denominator (2.1.8) immediately follows from the lim sup and lim inf bounds. To control the numerator, we note that the proof of Lemma 2.1.3 only uses the openness of $\mathcal{X}$ and can be replicated on all open $S \subset \mathcal{X}$. Similarly, the closedness of $\mathcal{X}$ is all that is used in the proof of Lemma 2.1.4. More detail can be found in the proof of [Hol08, Theorem III.17].

□

Remark 2.1.6. *In Chapter 3, we will be working with a space $\mathcal{X}$ that may not be Polish. For this we refer to the proof of Varadhan's Lemma in [Hol08, Theorem III.13]. Like the version in Theorem 2.1.5, the proof given here requires that the tilt F is continuous (the proof cannot be separated into semicontinuous versions) but also requires that F is bounded below. Crucially, the proof places no requirement on the topology other than F being continuous.* ◇

Remark 2.1.7. *Note that if F is bounded below, then the tail condition* (2.1.6) *holds necessarily. We will usually show this when calling upon Varadhan's Lemma.* ◇

Varadhan's Lemma tells us how to find large deviation principles for new measures that are defined by adding a tilt. However, for this to be useful we need a large deviation principle for the base measures in the first place. The Baldi and Gärtner-Ellis Theorems (Lemmas 2.1.14 and 2.1.10) allows us to derive large deviation principles from the logarithmic moment generating functions.

Definition 2.1.8. Let $\{P_N\}_{N\in\mathbb{N}}$ be a sequence of measures on the real vector space V with corresponding random variables $\{Z_N\}_{N\in\mathbb{N}}$, and V^* be the dual space to V. Then the *logarithmic moment generating function*, $\mathcal{L}_N : V^* \to [-\infty, +\infty]$, is given by

$$\mathcal{L}_N(s) := \log \mathbb{E}_N \left[\mathrm{e}^{\langle s, Z_N\rangle}\right], \qquad s \in V^*. \tag{2.1.11}$$

Then the *limiting logarithmic moment generating function* $\mathcal{L} : V^* \to [-\infty, +\infty]$ with rate r_N is defined as the limit superior

$$\mathcal{L}(s) := \limsup_{N\to\infty} \frac{1}{r_N} \mathcal{L}_N(r_N s), \qquad s \in V^*. \tag{2.1.12}$$

Let $\mathcal{L}^*$ denote the *Legendre transform* of $\mathcal{L}$:

$$\mathcal{L}^*(x) = \sup_{s\in V^*} \{\langle x, s\rangle - \mathcal{L}(s)\}, \qquad x \in V. \tag{2.1.13}$$

A point $x \in V$ is called *exposed* for $\mathcal{L}^*$ if there exists a point $t \in V^*$ such that

$$\langle x, t\rangle - \mathcal{L}^*(x) > \langle y, t\rangle - \mathcal{L}^*(y), \qquad \forall y \neq x. \tag{2.1.14}$$

Such a t is called (the normal to) an *exposing hyperplane* for x.

Definition 2.1.9. Let the set $\mathcal{D}_{\mathcal{L}} = \left\{s \in \mathbb{R}^K : \mathcal{L}(s) < +\infty\right\}$. A convex function $\mathcal{L} : \mathbb{R}^K \to (-\infty, +\infty]$ is *essentially smooth* if:

- $\operatorname{int}(\mathcal{D}_{\mathcal{L}}) \neq \varnothing$,
- $\mathcal{L}$ is differentiable on $\operatorname{int}(\mathcal{D}_{\mathcal{L}})$,
- $\mathcal{L}$ is *steep*, namely, $\lim_{n\to\infty} |\nabla\mathcal{L}(s_n)| = \infty$ whenever $\{s_n\}$ is a sequence in $\operatorname{int}(\mathcal{D}_{\mathcal{L}})$ converging to a boundary point of $\operatorname{int}(\mathcal{D}_{\mathcal{L}})$.

Here we present for convenience a combination of results from [DZ09] that will allow us to produce large deviation principles upon which we can build with Varadhan's Lemma.

Lemma 2.1.10. *Let $\{P_N\}_{N\in\mathbb{N}}$ be a sequence of measures on $\mathbb{R}^K$ with limiting logarithmic moment generating function $\mathcal{L}\colon \mathbb{R}^K \to [-\infty, \infty]$ such that $0 \in \operatorname{int}(\mathcal{D}_{\mathcal{L}})$. Then*

$$\limsup_{N\to\infty} \frac{1}{r_N} \log P_N(C) \leq -\inf_C \mathcal{L}^*, \qquad \forall C \subset \mathbb{R}^K \text{ closed}, \tag{2.1.15}$$

$$\liminf_{N\to\infty} \frac{1}{r_N} \log P_N(O) \geq -\inf_{O\cap\mathcal{F}} \mathcal{L}^*, \qquad \forall O \subset \mathbb{R}^K \text{ open}, \tag{2.1.16}$$

where $\mathcal{F}$ is the set of exposed points of $\mathcal{L}^*$ with an exposing hyperplane t such that $\mathcal{L}(t)$ is in fact a limit (rather than just a limit superior) and $\mathcal{L}(\gamma t) < \infty$ for some $\gamma > 1$.

If $\mathcal{L}$ is a limit everywhere, and is an essentially smooth, lower semicontinuous function, then the large deviation principle holds with the good rate function $\mathcal{L}^*$.

Proof. Baldi's Theorem [DZ09, Theorem 4.5.20] gives the bounds (2.1.15) and (2.1.16) if the sequence $\{P_N\}_{N\in\mathbb{N}}$ is exponentially tight (see Definition 2.1.12). The proof of the Gärtner-Ellis Theorem in [DZ09, Theorem 2.3.6] uses the Markov inequality to prove that in the finite dimensional space $\mathbb{R}^K$ the condition $0 \in \operatorname{int}(\mathcal{D}_{\mathcal{L}})$ implies exponential tightness. The large deviation principle clause follows from [DZ09, Theorem 2.3.6].

□

Remark 2.1.11. *The conditions required of Lemma 2.1.10 are not tight. Often we will not have $\mathcal{L}$ being a limit everywhere or not have essential smoothness. Nevertheless, we will be able to use the large deviation upper bound* (2.1.15) *and adapt the proofs of [DZ09] with details specific to our cases that allow us to derive the large deviation lower bound.* ◇

We will also want to have a result like Lemma 2.1.10 for infinite dimensional spaces. For this we make more direct use of Baldi's Theorem via the following lemma.

Definition 2.1.12. A sequence of measures $\{P_N\}_{N\in\mathbb{N}}$ is *exponentially tight* if for all $\gamma > 0$, there exists a compact set K_γ such that

$$\limsup_{N\to\infty} \frac{1}{r_N} \log P_N\left(K_\gamma^c\right) < -\gamma. \tag{2.1.17}$$

Definition 2.1.13. A function $f : X \to \mathbb{R}$ is *Gâteaux differentiable* if, for every $x, y \in X$, the function $f(x + ty)$ is differentiable with respect to t at $t = 0$.

Lemma 2.1.14. *Suppose $\{P_N\}_{N\in\mathbb{N}}$ is an exponentially tight sequence of measures on $\ell_1(\mathbb{R})$. Let $\mathcal{L}\colon \ell_\infty(\mathbb{R}) \to [-\infty, \infty]$ be the limiting cumulant generating function,* *and suppose that it is a limit and is finite for $t \in \ell_\infty(\mathbb{R})$. If $\mathcal{L}$ is Gâteaux differentiable, and lower semicontinuous on $\ell_\infty(\mathbb{R})$, then $\{P_N\}_{N\in\mathbb{N}}$ satisfies a large deviation principle with the good rate function $\mathcal{L}^*$.*

Proof. This is a corollary of Baldi's Theorem given by [DZ09, Corollary 4.5.27] for measures on general Banach spaces.

□

In order to find large deviation principles for restricted probability spaces, we will make use of the contraction principle.

Lemma 2.1.15 (**Contraction Principle**). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff topological spaces and $f : \mathcal{X} \to \mathcal{Y}$ a continuous function. Consider a good rate function $I : \mathcal{X} \to [0, \infty]$.*

1. *For each $y \in \mathcal{Y}$, define*

$$I'(y) = \inf\{I(x) : x \in \mathcal{X}, y = f(x)\}, \tag{2.1.18}$$

where $\inf \varnothing = +\infty$. Then I' is a good rate function on $\mathcal{Y}$.

2. *If I controls the large deviation principle associated with a family of probability measures $\{P_N\}_{N\in\mathbb{N}}$ on $\mathcal{X}$, then I' controls the large deviation principle associated with a family of probability measures $\{P_N \circ f^{-1}\}_{N\in\mathbb{N}}$ on $\mathcal{Y}$.*

Proof. This follows from [DZ09, Theorem 4.2.1].

□

2.2 Stationary Empirical Measures

Here we present the random object which our large deviation principles will aim to describe. In broad terms, the stationary empirical measure is a translation invariant empirical field of an instance of a marked point process.

Let us begin with these marks. The mark space is defined as

$$E_\Lambda^{(\mathrm{bc})} = \bigcup_{k\in\mathbb{N}} \mathcal{C}_{k,\Lambda}^{(\mathrm{bc})}, \qquad \mathrm{bc} \in \{\varnothing, \mathrm{per}, \mathrm{Dir}\}, \tag{2.2.1}$$

where, for $k \in \mathbb{N}$, we denote by $\mathcal{C}_k = \mathcal{C}_{k,\Lambda}^{(\varnothing)}$ the set of continuous functions $f\colon [0, k\beta] \to \mathbb{R}^d$ satisfying $f(0) = f(k\beta)$, equipped with the topology of uniform convergence (see Definition 2.2.5). In addition, $\mathcal{C}_{k,\Lambda}^{(\mathrm{Dir})}$, (respectively $\mathcal{C}_{k,\Lambda}^{(\mathrm{per})}$) is the space of continuous functions in Λ (respectively on the torus $\Lambda = (\mathbb{R}/L\mathbb{Z})^d$) with time horizon $[0, k\beta]$. We sometimes call the marks *cycles*. By $\ell\colon E_\Lambda^{(\mathrm{bc})} \to \mathbb{N}$ we denote the canonical map defined by $\ell(f) = k$ if $f \in \mathcal{C}_{k,\Lambda}^{(\mathrm{bc})}$. We call $\ell(f)$ the *length* of $f \in E_\Lambda^{(\mathrm{bc})}$. When dealing with the empty boundary condition, we sometimes drop the superscript $\varnothing$.

We consider the set Ω of counting measures on $\mathbb{R}^d \times E$ for the empty boundary condition (respectively on $\Lambda \times E_\Lambda^{(\mathrm{bc})}$ for $\mathrm{bc} \in \{\mathrm{per}, \mathrm{Dir}\}$). These consist of a locally finite set $\xi = \xi(\omega) \subset \mathbb{R}^d$ of anchors, and to each anchor $x \in \xi$ we attach a mark $f_x \in E_\Lambda^{(\mathrm{bc})}$ satisfying $f_x(0) = x$. Hence, a configuration is described by the counting measure

$$\omega = \sum_{x \in \xi} \delta_{(x, f_x)}. \tag{2.2.2}$$

Given a configuration $\omega \in \Omega$ and a box Λ, a natural object is the *anchor number*

$$N_\Lambda(\omega) = \#\{x \in \xi \cap \Lambda\}. \tag{2.2.3}$$

A related object is the *particle number*

$$N_\Lambda^{(\ell)}(\omega) = \sum_{k \in \mathbb{N}} k \# \{x \in \xi \cap \Lambda : \ell(f_x) = k\}. \tag{2.2.4}$$

Both of these can also be defined for bounded subsets of $\mathbb{R}^d$ other than Λ, such as the unit cube $U = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

We now introduce three marked Poisson point processes for the three boundary conditions. The one for the empty boundary condition will later serve as a reference process and is introduced separately first. The remaining two processes are defined in Appendix A.

Definition 2.2.1. (**Reference process**) Consider on $\mathcal{C} = \mathcal{C}_1$ the canonical Brownian bridge measure

$$\boldsymbol{\mu}_{x,y}^{(\varnothing,\beta)}(A) = \boldsymbol{\mu}_{x,y}^{(\beta)}(A) = \mathbb{P}_x(\mathbb{1}\{B \in A\}\, \delta_y(B_\beta)), \qquad A \subset \mathcal{C} \text{ measurable}. \tag{2.2.5}$$

This (non-normalised) measure is given by the kernel of a Brownian motion in $\mathbb{R}^d$ with generator Δ, starting at x and terminating at y at parameter time β. It follows that $\boldsymbol{\mu}_{x,y}^{(\beta)}$ is a regular Borel measure on $\mathcal{C}$ with total mass equal to the Gaussian density,

$$\boldsymbol{\mu}_{x,y}^{(\beta)}(\mathcal{C}) = g_\beta(x,y) = \mathbb{P}_x(\delta_y(B_\beta)) = \frac{1}{(4\pi\beta)^{\frac{d}{2}}} \exp\left(-\frac{1}{4\beta}|x-y|^2\right). \tag{2.2.6}$$

We write $\mathbb{P}_{x,y}^{(\beta)} = \boldsymbol{\mu}_{x,y}^{(\beta)}/g_\beta(x,y)$ for the normalised Brownian bridge measure on $\mathcal{C}$.

Let

$$\omega_{\mathrm{P}} = \sum_{x\in\xi_{\mathrm{P}}} \delta_{(x,B_x)}, \tag{2.2.7}$$

be a Poisson point process on $\mathbb{R}^d \times E$ with intensity measure equal to ν, whose projection onto $\mathbb{R}^d \times \mathcal{C}_k$ is equal to

$$\nu_k(\mathrm{d}x,\mathrm{d}f) = \frac{1}{k}\mathrm{e}^{\beta\alpha k}\mathrm{Leb}(\mathrm{d}x) \otimes \boldsymbol{\mu}_{x,x}^{(k\beta)}(\mathrm{d}f), \qquad k \in \mathbb{N}. \tag{2.2.8}$$

Remark 2.2.2. *Note that here we denote the chemical potential with α. Throughout this thesis we shall have some freedom over whether we treat the chemical potential as part of the underlying reference measure or as part of the energy terms. We shall let α refer to the reference measure chemical potential, but shall require $\alpha \leq 0$ for it be well-defined. We shall denote the energetic chemical potential with μ. If we are considering what we will call a 'stabilised' model, then we shall be able to consider any $\mu \in \mathbb{R}$, but otherwise we will only be able to have well-defined models for $\alpha + \mu \leq 0$.*

Alternatively, we can conceive ω_{P} as a marked Poisson point process on $\mathbb{R}^d$, based on some Poisson point process ξ_{P} on $\mathbb{R}^d$, and a family $(B_x)_{x\in\xi_{\mathrm{P}}}$ of i.i.d. marks, given ξ_{P}. The intensity of ξ_{P} is

$$\overline{q}^{(\alpha)} = \sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\beta\alpha k}, \qquad \text{with} \quad q_k = \frac{1}{k} g_{\beta k}(x,x) = \frac{1}{(4\pi\beta)^{d/2} k^{1+d/2}}, \qquad k \in \mathbb{N}. \tag{2.2.9}$$

Conditionally given ξ_{P}, the length $\ell(B_x)$ is an $\mathbb{N}$-valued random variable with distribution $(q_k \mathrm{e}^{\beta\alpha k}/\overline{q}^{(\alpha)})_{k\in\mathbb{N}}$, and, given $\ell(B_x) = k$, B_x is in distribution equal to a

Brownian bridge with time horizon $[0, k\beta]$, starting and ending at x.

Let $\mathtt{Q}_\alpha$ denote the distribution of ω_{P} and denote by $\mathtt{E}_\alpha$ the corresponding expectation (respectively, write $\mathtt{Q}_\alpha^{(\mathrm{bc})}$ for $\mathrm{bc} \in \{\mathrm{per}, \mathrm{Dir}\}$). Hence, $\mathtt{Q}_\alpha$ is a probability measure on the set Ω of all locally finite counting measures on $\mathbb{R}^d \times E$. Note that our reference process is a countable superposition of Poisson point processes, and - as long as $\overline{q}^{(\alpha)}$ is finite - this reference process is a Poisson point process as well.

Definition 2.2.3. (**Stationary Empirical Measure**) For any $\xi \subset \mathbb{R}^d$ and for any centred box $\Lambda \subset \mathbb{R}^d$, let $\xi_{(\Lambda)}$ be the Λ-periodic continuation of $\xi \cap \Lambda$. Analogously, we define the Λ-periodic continuation of the restriction of the configuration ω to Λ as

$$\omega_{(\Lambda)} = \sum_{z \in \mathbb{Z}^d} \sum_{x \in \xi \cap \Lambda} \delta_{(x+Lz, f_x)}, \qquad \text{if } \omega = \sum_{x \in \xi} \delta_{(x, f_x)} \in \Omega, \tag{2.2.10}$$

where L is the side length of the centred cube Λ. The shift operator $\theta_y : \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\theta_y(x) = x - y$, and it extends naturally to a shift operator on marked configurations with

$$\theta_y(\omega) = \sum_{x \in \xi} \delta_{(x-y, f_{x-y})} = \sum_{x \in \xi - y} \delta_{(x, f_x)}, \qquad \text{for } \omega = \sum_{x \in \xi} \delta_{(x, f_x)}. \tag{2.2.11}$$

We denote by $\mathcal{P}_\theta$ the set of all shift-invariant probability measures, P, on Ω such that there exists a number $z(P) < +\infty$ such that $\mathbb{E}_P[N_\Delta] = z(P)|\Delta|$ for all Borel $\Delta \subset \mathbb{R}^d$. The *stationary empirical field* is the map $\mathfrak{R}_{\Lambda,\cdot} : \Omega \to \mathcal{P}_\theta$ given by

$$\mathfrak{R}_{\Lambda,\omega} = \frac{1}{|\Lambda|} \int_\Lambda \mathrm{d}y \delta_{\theta_y(\omega_{(\Lambda)})}. \tag{2.2.12}$$

From its construction it is clear that $\mathfrak{R}_{\Lambda,\omega}$ is shift invariant, and from a minor variation on Lemma 3.1.11, we can show that $z(\mathfrak{R}_{\Lambda,\omega}) = \frac{1}{|\Lambda|} N_\Lambda(\omega)$ and is finite (since ξ is a locally finite set). Given a distribution P on Ω, we will be interested in deriving large deviation principles for $P \circ (\mathfrak{R}_{\Lambda,\cdot})^{-1}$ on $\mathcal{P}_\theta$.

The large deviations of the stationary empirical field of the reference process has already been studied by Georgii and Zessin in [GZ93]. We present the rate function and the topology of the large deviation principle before giving the result itself.

Definition 2.2.4. For probability measures μ, ν on some measurable space, we write

$$H(\mu|\nu) = \begin{cases} \int f \log f \mathrm{d}\nu, & \text{if } f = \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \text{ exists,} \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2.13}$$

for the *relative entropy* of μ with respect to ν. Then we can set

$$I_\alpha(P) = \lim_{N\to\infty} \frac{1}{|\Lambda_N|} H\left(P_{\Lambda_N} | \mathtt{Q}_{\alpha,\Lambda_N}\right), \qquad P \in \mathcal{P}_\theta, \tag{2.2.14}$$

where we write P_Λ for the projection of P to Λ, that is, the image measure of P under

$$\omega \mapsto \omega|_\Lambda = \sum_{x\in\xi\cap\Lambda} \delta_{(x,f_x)} \qquad \text{for } \omega = \sum_{x\in\xi} \delta_{(x,f_x)}. \tag{2.2.15}$$

Definition 2.2.5. A measurable function $g : \Omega \to \mathbb{R}$ is called *local* if it depends only on the restriction of ω to some bounded open cube, and it is called *tame* if $|g| \le c(1 + N_\Lambda)$ for some bounded open cube Λ and some constant $c \in \mathbb{R}_+$. We endow the space $\mathcal{P}_\theta$ with the topology $\tau_\mathcal{L}$ of *local convergence*, defined as the smallest topology on $\mathcal{P}_\theta$ such that the mappings $P \mapsto \langle P, g\rangle$ are continuous for any $g \in \mathcal{L}$, where $\mathcal{L}$ denotes the linear space of all local tame functions. These maps are defined on $\mathcal{P}_\theta$ because for each $P \in \mathcal{P}_\theta$ the expectations $\langle P, N_\Delta\rangle$ are finite for all Borel $\Delta \subset \mathbb{R}^d$.

Theorem 2.2.6 (**LDP for non-interacting Stationary Empirical Measure**). *For $\alpha \le 0$, the measures $\mathtt{Q} \circ (\mathfrak{R}_{\Lambda,\cdot})^{-1}$ satisfy, as $L \to \infty$, a large deviations principle on $\mathcal{P}_\theta$ with the topology $\tau_\mathcal{L}$, with rate $|\Lambda_L|$ and rate function $I_\alpha : \mathcal{P}_\theta \to [0, \infty]$ defined in* (2.2.14). *The function I_α is affine and lower $\tau_\mathcal{L}$-semicontinuous and has $\tau_\mathcal{L}$-compact level sets.*

Proof. This is [GZ93, Theorem 3.1].

□

Now we introduce interactions into the picture. We call the restriction of a mark f_x to the interval $[i\beta, (i + 1)\beta)]$ with $i \in \{0, \dots, \ell(f_x) - 1\}$ a leg of the mark.

Imagine that each leg of each mark has a particle moving along it at parameter speed, and that these particles interact via a pair-potential v. In particular, note that marks in general have a self-interaction. Given a box-restricted configuration $\omega|_\Lambda$, the Hamiltonian H_Λ then gives the total energy of the particles particles within this box, and probabilities are adjusted via the Boltzmann factor of this energy.

Definition 2.2.7. We define the *anchor interaction* between x and y (allowing for $x = y$) as

$$T_{x,y}(\omega) = \frac{1}{2} \sum_{i=0}^{\ell(f_x)-1} \sum_{j=0}^{\ell(f_y)-1} \mathbb{1}_{\{(x,i)\neq(y,j)\}} \int_0^\beta v(|f_x(i\beta + s) - f_y(j\beta + s)|)\, \mathrm{d}s,$$

$$\omega \in \Omega, x, y \in \xi. \quad (2.2.16)$$

Then the *Hamiltonian* $H_\Lambda : \Omega \to [-\infty, +\infty]$ is given by

$$H_\Lambda(\omega) = \sum_{x,y\in\xi\cap\Lambda} T_{x,y}(\omega), \qquad \text{where } \omega = \sum_{x\in\xi} \delta_{(x,f_x)} \in \Omega. \quad (2.2.17)$$

Definition 2.2.8. To enable us to consider any chemical potential, we also introduce a stabilisation term. Given $\mu > 0$, let $U_\mu : [0, +\infty] \to \mathbb{R} \cup \{+\infty\}$ be a continuous function such that $U_\mu(x) \gg x$ as $x \to \infty$ and $U_\mu(x) - \mu x$ is non-decreasing in x. Then we introduce the *stabilisation energy*, $V_{\Lambda,\mu}$:

$$V_{\Lambda,\mu}(\omega) = |\Lambda| U_\mu \left(\frac{N_\Lambda^{(\ell)}(\omega)}{|\Lambda|} \right), \quad (2.2.18)$$

where $N_\Lambda^{(\ell)}$ is the particle number defined in (2.2.4). If we wish to consider a non-positive chemical potential, then we will be free to ignore this stabilisation because $-\mu x$ is already non-decreasing and bounded below. In fact, we will be able to describe it only using the reference measure chemical potential. For positive chemcial potentials, this stabilisation is required to ensure that the energetic chemical potential energy does not lead to arbitrary negative total energies.

These interactions are then incorporated into measures on Ω as follows:

$$\mathtt{Q}^{(H)}_{\Lambda,\mu,\alpha}(\mathrm{d}\omega) = \frac{\mathrm{e}^{\left(\beta\mu N^{(\ell)}_\Lambda - \beta V_{\Lambda,\mu} - H_\Lambda\right)(\omega)}}{\mathtt{E}^{(\mathrm{bc})}_\alpha\left[\mathrm{e}^{\beta\mu N^{(\ell)}_\Lambda - \beta V_{\Lambda,\mu} - H_\Lambda}\right]}\mathtt{Q}^{(\mathrm{bc})}_\alpha(\mathrm{d}\omega), \quad \alpha \le 0 \tag{2.2.19}$$

Lemma 2.2.9. *Fix $\beta \in (0,\infty)$. Let $v : \mathbb{R}_+ \to \mathbb{R}\cup\{+\infty\}$ be measurable and bounded from below, and let $\Lambda \subset \mathbb{R}^d$ be measurable with finite volume (assumed to be a torus for periodic boundary condition). Then, for any* $\mathrm{bc} \in \{\varnothing, \mathrm{per}, \mathrm{Dir}\}$,

$$Z^{(\mathrm{bc})}_\Lambda(\beta,\mu,\alpha) = \mathrm{e}^{|\Lambda|\overline{q}^{(\mathrm{bc},\alpha)}_\Lambda}\mathtt{E}^{(\mathrm{bc})}_\alpha\left[\mathrm{e}^{\beta\mu N^{(\ell)}_\Lambda - \beta V_{\Lambda,\mu} - H_\Lambda}\right]. \tag{2.2.20}$$

Proof. This follows from [ACK11, Proposition 1.1], with the omission of the canonical ensemble indicator function. This produces no extra complications.

□

Remark 2.2.10. *The formula* (2.2.20) *has also been obtained for Poisson loop processes on graphs, see [AV19].* ◇

Lemma 2.2.11. *Suppose $v \equiv 0$, $V_{\Lambda,\mu} \equiv 0$ and $\{\Lambda_N\}_{N\in\mathbb{N}}$ are measurable boxes in $\mathbb{R}^d$ such that $|\Lambda_N| \to \infty$. Also suppose $\alpha \le 0$ and* $\mathrm{bc} \in \{\varnothing, \mathrm{Dir}\}$ *(or $\alpha < 0$ and* $\mathrm{bc} = \mathrm{per}$*). Then for $\beta > 0$ the non-interacting thermodynamic pressure is given by*

$$p(\beta,\alpha) := \lim_{N\to\infty}\frac{1}{\beta|\Lambda_N|}\log Z^{(\mathrm{bc})}_{\Lambda_N}(\beta,\alpha) \tag{2.2.21}$$
$$= \frac{1}{\beta}\sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\beta\alpha k}. \tag{2.2.22}$$

Proof. The finite $|\Lambda_N|$ pressure follows from Lemma 2.2.9. The limit is trivial for $\mathrm{bc} = \varnothing$, and follows from the analysis in Appendix A if $\mathrm{bc} \in \{\mathrm{Dir}, \mathrm{per}\}$. In particular, $Z^{(\mathrm{per})}_{\Lambda_N}(\beta, 0) = +\infty$.

□

2.3 Empirical Cycle Counts

Now we reduce our resolution from the full detail of Ω to the countable set of random variables

$$\mathcal{N}_{\Lambda,k}(\omega) = \#\left\{x \in \xi \cap \Lambda \colon \ell(f_x) = k\right\}, \qquad k \in \mathbb{N}. \tag{2.3.1}$$

These are the number of anchors in Λ with mark length equal to k, and under the reference distribution induced by $\mathtt{Q}_\alpha$ they are independent Poisson random variables with respective means $|\Lambda| q_k \mathrm{e}^{\beta\alpha k}$.

Definition 2.3.1. (**Empirical Cycle Count**) The *empirical cycle count* is the empirical density of numbers of cycles of all lengths,

$$\boldsymbol{\lambda}^{(\Lambda)}(\omega) = \left(\boldsymbol{\lambda}_k^{(\Lambda)}(\omega)\right)_{k\in\mathbb{N}} = \left(\frac{1}{|\Lambda|}\mathcal{N}_{\Lambda,k}(\omega)\right)_{k\in\mathbb{N}}, \tag{2.3.2}$$

taking values in $\ell_1(\mathbb{R}_+)$.

The *empirical total density* can then be given as the following unbounded linear transformation of the empirical cycle count:

$$\boldsymbol{\rho}_\Lambda(\omega) = \frac{1}{|\Lambda|} N_\Lambda^{(\ell)}(\omega) = \sum_{k\in\mathbb{N}} k \boldsymbol{\lambda}_k^{(\Lambda)}(\omega). \tag{2.3.3}$$

Suppose that we have the sequence

$$\{\lambda_k\}_{k\in\mathbb{N}} \in \mathcal{M}_\Lambda := \left\{\lambda \in \mathbb{R}_+^{\mathbb{N}} : \sum_{k\in\mathbb{N}} k\lambda_k < \infty, |\Lambda|\lambda_k \in \mathbb{N}_0\right\} \subset \ell_1(\mathbb{R}_+). \tag{2.3.4}$$

Then, using the independence and the Poisson nature of our reference process, we compute the probability

$$\mathtt{Q}_\alpha\left(\boldsymbol{\lambda}^{(\Lambda)}(\omega_P) = (\lambda_k)_{k\in\mathbb{N}}\right) = \prod_{k\in\mathbb{N}} \frac{\mathrm{e}^{-|\Lambda| q_k^{(\mathrm{bc})}\mathrm{e}^{\beta\alpha k}} \left(|\Lambda| q_k^{(\mathrm{bc})}\mathrm{e}^{\beta\alpha k}\right)^{|\Lambda|\lambda_k}}{(|\Lambda|\lambda_k)!}. \tag{2.3.5}$$

The induced distribution on $\ell_1(\mathbb{R}_+)$ is denoted

$$\nu_{N,\mu}^{(\mathrm{bc})} := \mathtt{Q}_\alpha^{(\mathrm{bc})} \circ \left(\boldsymbol{\lambda}^{(\Lambda_N)}\right)^{-1}, \tag{2.3.6}$$

which in fact has support $\mathcal{M}_\Lambda$.

2.3.1 Interactions as a Function of Cycle Count

We shall now define our various interaction models. We are not able to consider the pair potential that we study in the context of the stationary empirical field because this is inexpressible in terms of our low-resolution cycle counts. However, there are approximations of interactions that can be written in terms of cycle counts.

Incorporating interactions into statistical mechanical models is often a difficult task. One common approach is to use a mean field approximation. Working in momentum space, [Lew86] gives the following heuristic derivation of two such approximations for the Bose gas.

Consider the full quantum mechanical Hamiltonian H^ϕ_Λ for a system of bosons in a cube $\Lambda \subset \mathbb{R}^d$ interacting through a pair-potential $\phi \in L^1\left(\mathbb{R}^d\right)$. This Hamiltonian is an operator on the symmetric Fock space $\mathcal{F}_+ = \bigoplus_{m=0}^\infty L^2_+\left(\Lambda^{\otimes m}\right)$ constructed on symmetric Hilbert spaces, given by

$$\mathcal{H}^\phi_\Lambda = \bigoplus_{m=0}^{\infty} h_{\Lambda,m} + U^\phi = \bigoplus_{m=0}^{\infty} \left(h_{\Lambda,m} + u^\phi_m\right), \tag{2.3.7}$$

$$h_{\Lambda,m} = -\sum_{i=1}^{m} \Delta_{\Lambda,i} \tag{2.3.8}$$

$$u^\phi_m = \sum_{1\leq i<j\leq m} \phi\left(x_i - x_j\right) \tag{2.3.9}$$

where $-\Delta_{\Lambda,i}$ is the kinetic energy of the i^{th} particle in $\Lambda^{\otimes m}$, and U^ϕ is the interaction operator given by its action on the m-particle Hilbert subspaces of $\mathcal{F}_+\left(\Lambda\right)$.

This can be written in terms of annihilation and creation operators: $a\left(k\right)$ and $a^\dagger\left(k\right)$. The index k denotes the wave number (corresponding to momentum) and the operators satisfy the canonical commutation relations $\left[a\left(k_1\right), a^\dagger\left(k_2\right)\right] = \delta_{k_1,k_2}$.

Then

$$\mathcal{H}_\Lambda^\phi = \sum_k \epsilon_\Lambda(k)\, a^\dagger(k)\, a(k) + \frac{1}{2|\Lambda|} \sum_{k_1} \sum_{k_2} \sum_q \hat{\phi}_\Lambda(q)\, a^\dagger(k_1)\, a^\dagger(k_2)\, a(k_2 - q)\, a(k_1 + q)\,, \tag{2.3.10}$$

where

$$\hat{\phi}_\Lambda(q) = \int_\Lambda \phi(y)\, \mathrm{e}^{iq\cdot y} \mathrm{d}y. \tag{2.3.11}$$

As usual, the operator $n(k) = a^\dagger(k)\, a(k)$ is interpreted as the occupation number of the eigenstate of the kinetic energy operator with eigenvalue $\epsilon_\Lambda(k)$.

The main approximation made by [Lew86] is to neglect the terms in the Hamiltonian which are not diagonal in $n(k)$. In particular, this forces the Hamiltonian to be space homogeneous. If we also assume that $\hat{\phi}_\Lambda$ can be approximated with the full Fourier transform $\hat{\phi}$ (replace Λ with $\mathbb{R}^d$), the diagonal part U_d^ϕ of the interaction operator can be written as

$$U_d^\phi = U_1^\phi + U_2^\phi + U_3^\phi, \tag{2.3.12}$$

$$U_1^\phi = \frac{\hat{\phi}(0)}{2|\Lambda|}\left(\mathcal{N}^2 - \mathcal{N}\right), \tag{2.3.13}$$

$$U_2^\phi = \frac{\hat{\phi}(0)}{2|\Lambda|}\left(\mathcal{N}^2 - \sum_k n(k)^2\right), \tag{2.3.14}$$

$$U_3^\phi = \frac{1}{2|\Lambda|} \sum_{k_1} \sum_{k_2} \left(\hat{\phi}(k_1 - k_2) - \hat{\phi}(0)\right) n(k_1)\, n(k_k)\,, \tag{2.3.15}$$

where $\mathcal{N} = \sum_k n(k)$ is the total number operator. Now, in the thermodynamic limit, the term $-\frac{\hat{\phi}(0)}{2|\Lambda_N|}\mathcal{N}$ in U_1^ϕ makes no meaningful contribution to the energy *density*, and so we may neglect it. Factoring it in would only change every chemical potential by a term proportional to $\hat{\phi}(0)$.

Now we can arrive at two different models by making one of two approximations. First, let us replace ϕ with (approximately) its spatial average, $\widetilde{\phi}_\Lambda$, given by

$$\widetilde{\phi}_\Lambda(y) = \frac{1}{|\Lambda|}\int_{\mathbb{R}^d} \phi(y)\, \mathrm{d}y = \frac{\hat{\phi}(0)}{|\Lambda|}. \tag{2.3.16}$$

Then we have $\hat{\hat{\phi}}_\Lambda (k_1 - k_2) = \hat{\phi}(0)\, \delta_{k_1,k_2}$ and U_2^ϕ cancels with U_3^ϕ. Thus we get the *mean field model* with interaction energy

$$U = \frac{\hat{\phi}(0)}{2|\Lambda|} \mathcal{N}^2. \tag{2.3.17}$$

This interaction approximation can be understood classically: it arises in an "index of refraction" approximation. [HY57] asks us to imagine each particle moving through the system as if the latter were a uniform optical medium. Then we would expect each particle to have an increment in energy proportional to $n^2 - 1$, where n is the index of refraction. For a medium of low density, a classical result of Lagrange states that

$$n^2 - 1 \propto \frac{\mathcal{N}}{|\Lambda|}, \tag{2.3.18}$$

and therefore one expects the energy increment of the total system to be proportional to $\mathcal{N}^2/|\Lambda|$.

[HY57] also gives an argument that that the potential should be proportional to the scattering length ξ. At very low energies, the interaction between particles is shape independent, and characterised by only a single parameter, the scattering length ξ. Therefore we can replace the actual inter-particle potential by a square well of such radius and depth that it gives rise to the same scattering length. To a single particle which moves through the system, the square wells presented by the remaining $N - 1$ particles may overlap to form a constant potential, whence the medium-like behaviour of the system. The depth of this effective constant potential is then proportional to the scattering length ξ. Note that a positive ξ (and a positive ϕ) would correspond to a repulsive field, and a negative value wouldE correspond to an attractive field.

Now instead suppose that ϕ is very short range - akin to a hard sphere repulsion - and thus $\hat{\phi}(q)$ is independent of q. Hence U_3^ϕ vanishes and we get the *Huang-Lang-Luttinger hard-sphere model* with interaction operator

$$U = \frac{\hat{\phi}(0)}{|\Lambda|} \left(\mathcal{N}^2 - \frac{1}{2} \sum_k n(k)^2 \right). \tag{2.3.19}$$

Introduced in [HYL57], this interaction is purely quantum mechanical in origin and is a first order approximation of the effect of Bose-Einstein statistics on the interaction energy of a system of bosons interacting through hard-sphere repulsion.

Now whilst the mean field and HYL energies have been derived from a momentum perspective, we can translate some parts into our cycle count framework. Clearly we can replace the operators $\mathcal{N}$ with our loop representation of the total physical particle number $N_\Lambda^{(\ell)}$. However, although this is sufficient for the mean field model, we would want to find a way to express the $n\left(k\right)$ with our cycle type counts to get the proper HYL model. Although there is a duality between momentum and position (for example in the famous Heisenburg's Uncertainty Principle) that would suggest long cycles may be related in some way to low momenta, this falls a long way short of our requirements. Instead we will take a more abstract view of their role in the HYL energy. We will view $n\left(k\right)$ as the number of physical particles of designated type k, given by $k\mathcal{N}_{\Lambda,k}$ in our cycle model.

2.3.2 Cycle Mean Field Model

This first class of model is loosely based on the mean field interaction energy arrived at in (2.3.17). Here we replace $\mathcal{N}$ with the total anchor number N_Λ, and $a > 0$ takes the role of $\hat{\phi}\left(0\right)$.

Definition 2.3.2. Define the cycle-mean-field (CM) model,

$$H^{(\text{CM})}(x) = \frac{a}{2}\left(\sum_{k=1}^{\infty} x_k\right)^2, \qquad x \in \ell_1(\mathbb{R}_+), \tag{2.3.20}$$

$$\nu_{N,\alpha}^{(\text{CM})}(\mathrm{d}x) = \frac{\exp(-|\Lambda_N|\beta H^{(\text{CM})}(x))}{Z_N^{(\text{CM})}(\beta,\alpha)}\nu_{N,\alpha}^{(\text{bc})}(\mathrm{d}x), \quad \alpha \le 0, \tag{2.3.21}$$

with partition function

$$Z_N^{(\text{CM})}(\beta,\alpha) = \mathbb{E}_{\nu_{N,\alpha}^{(\text{bc})}}\left[\mathrm{e}^{-|\Lambda_N|\beta H^{(\text{CM})}(x)}\right] = \mathrm{E}_{N,\alpha}^{(\text{bc})}\left[\exp\left\{-\frac{\beta a}{2|\Lambda_N|}\left(N_{\Lambda_N}\right)^2\right\}\right]. \tag{2.3.22}$$

Remark 2.3.3. *Note that for the CM model we keep the chemical potential within the base measure. This preserves the continuity of the Hamiltonian.* $\diamond$

This is a relatively uninteresting model: it does nothing to distinguish between different marks, and so loses perhaps the most interesting aspect of the framework. Nevertheless, it serves as a useful toy model that we will use to demonstrate how we would like to proceed if the difficulties that arise in our later model were absent.

2.3.3 Particle Mean Field and Generalised Particle Mean Field Models

Here we enact the true physical mean field model. We replace $\mathcal{N}$ with the total physical particle number $N^{(\ell)}_{\Lambda}$, and $a > 0$ takes the role of $\hat{\phi}\,(0)$. For readability, we define

$$D : \ell_1\left(\mathbb{R}_+\right) \to [0, +\infty]\,, \qquad x \mapsto \sum_{j=1}^{\infty} j x_j. \tag{2.3.23}$$

Definition 2.3.4. Define the particle-mean-field (PM) model,

$$H^{(PM)}_{\mu}(x) = -\mu D\left(x\right) + \frac{a}{2} D\left(x\right)^2\,, \qquad x \in \ell_1(\mathbb{R}_+), \tag{2.3.24}$$

$$\nu^{(PM)}_{N,\mu,\alpha}(\mathrm{d}x) = \frac{\exp(-|\Lambda_N|\beta H^{(PM)}_{\mu}(x))}{Z^{(PM)}_{N}(\beta,\mu,\alpha)} \nu^{(\mathrm{bc})}_{N,\alpha}(\mathrm{d}x), \quad \alpha \leq 0, \mu \in \mathbb{R}, \tag{2.3.25}$$

with partition function

$$Z^{(PM)}_{N}(\beta,\mu,\alpha) = \mathbb{E}_{\nu^{(\mathrm{bc})}_{N,\alpha}}\left[\mathrm{e}^{-|\Lambda_N|\beta H^{(PM)}_{\mu}(x)}\right] = \mathrm{E}^{(\mathrm{bc})}_{N,\alpha}\left[\exp\left\{\beta\mu N^{(\ell)}_{\Lambda_N} - \frac{\beta a}{2|\Lambda_N|}\left(N^{(\ell)}_{\Lambda_N}\right)^2\right\}\right]. \tag{2.3.26}$$

To emphasise the properties of $H^{(PM)}_{\mu}$ that make it amenable, we introduce a generalised version.

Definition 2.3.5. Let $G : [0, +\infty] \to \mathbb{R} \cup \{+\infty\}$ be continuous and bounded below. Then we can define the generalised particle mean field model (GPM),

$$\nu^{(GPM)}_{N}(\mathrm{d}x) = \frac{\exp(-|\Lambda_N|\beta G \circ D(x))}{Z^{(GPM)}_{N}} \nu^{(\mathrm{bc})}_{N,\alpha}, \tag{2.3.27}$$

with partition function

$$Z_N^{(GPM)} = \mathbb{E}_{\nu_{N,\alpha}^{(\mathrm{bc})}} \left[\mathrm{e}^{-|\Lambda_N|\beta G \circ D} \right] = \mathtt{E}_{N,\alpha}^{(\mathrm{bc})} \left[\exp \left\{ -|\Lambda_N| \beta G \left(\frac{N_{\Lambda_N}^{(\ell)}}{|\Lambda_N|} \right) \right\} \right]. \tag{2.3.28}$$

Remark 2.3.6. *The PM model is a special case of the GPM model with* $G : x \mapsto -\mu x + \frac{a}{2} x^2$. ◇

Remark 2.3.7. *Note the similarities between the GPM model and the stabilising energy we introduced for the level-3 description. They both serve the purpose that they allow us to consider the whole chemical potential space (allowing* $\alpha + \mu > 0$*). Our GPM here is actually slightly more general because we don't require that* G *is non-decreasing. We will be able to overcome this extra challenge because we have a bit more information about the non-interacting rate function than we did for the level-3 description.* ◇

2.3.4 Cycle space Huang-Yang-Luttinger Models

We now arrive at the main focus of this thesis. The mean field models can be expressed solely in terms of the total anchor or physical particle number, but the HYL models that we now introduce require the extra detail that the empirical cycle count gives us.

By taking the full true HYL and replacing the the momentum eigenstate numbers $n(k)$ with the physical particle cycle type occupation numbers $k\mathcal{N}_{\Lambda,k}$, we get our first HYL model, which we call the *full cycle HYL* model (or FCH model).

Definition 2.3.8. For any $a \geq b > 0$ and any $\alpha \leq 0, \mu \in \mathbb{R}$, define the FCH-model by

$$H_\mu^{(FCH)}(x) = -\mu D(x) + \frac{a}{2} D(x)^2 - \frac{b}{2} \sum_{k=1}^{\infty} k^2 x_k^2, \qquad x \in \ell_1(\mathbb{R}_+), \tag{2.3.29}$$

$$\nu_{N,\mu,\alpha}^{(FCH)}(\mathrm{d}x) = \frac{\exp(-\beta |\Lambda_N| H_\mu^{(FCH)}(x))}{Z_N^{(FCH)}(\beta, \mu, \alpha)} \nu_{N,\alpha}^{(\mathrm{bc})}(\mathrm{d}x), \tag{2.3.30}$$

with partition function

$$\begin{aligned} Z_N^{(FCH)}(\beta,\mu,\alpha) &= \mathbb{E}_{\nu_{N,\alpha}^{(\mathrm{bc})}}\left[\mathrm{e}^{-|\Lambda_N|\beta H_\mu^{(FCH)}(x)}\right] \\ &= \mathtt{E}_{N,\alpha}^{(\mathrm{bc})}\left[\exp\left\{\beta\mu N_{\Lambda_N}^{(\ell)} - \frac{\beta a}{2|\Lambda_N|}\left(N_{\Lambda_N}^{(\ell)}\right)^2 + \frac{\beta b}{2|\Lambda_N|}\sum_{k=1}^{\infty} k^2 \mathcal{N}_{\Lambda_N,k}^2\right\}\right]. \end{aligned} \tag{2.3.31}$$

In Chapter 6, we will consider a slightly different version of the HYL model, which we will call the *partial cycle HYL* model (or PCH model). In this version, instead of effecting all cycle types, the b counter-term only effects a subset of types that varies as we take the thermodynamic limit.

Definition 2.3.9. Let us consider a sequence of boxes $\{\Lambda_N\}_{N\in\mathbb{N}}$, with $\{A_N\}_{N\in\mathbb{N}}$ being a sequence of subsets of $\mathbb{N}$. For any $a \geq b > 0$ and any $\alpha \leq 0, \mu \in \mathbb{R}$, define the PCH-model by

$$H_{\mu,N}^{(PCH)}(x) = -\mu D(x) + \frac{a}{2}D(x)^2 - \frac{b}{2}\sum_{k\in A_N} k^2 x_k^2, \qquad x \in \ell_1(\mathbb{R}_+), \tag{2.3.32}$$

$$\nu_{N,\mu,\alpha}^{(PCH)}(\mathrm{d}x) = \frac{\exp(-\beta|\Lambda_N|H_{\mu,N}^{(PCH)}(x))}{Z_N^{(PCH)}(\beta,\mu,\alpha)}\nu_{N,\alpha}^{(\mathrm{bc})}(\mathrm{d}x), \tag{2.3.33}$$

with partition function

$$\begin{aligned} Z_N^{(PCH)}(\beta,\mu,\alpha) &= \mathbb{E}_{\nu_{N,\alpha}^{(\mathrm{bc})}}\left[\mathrm{e}^{-|\Lambda_N|\beta H_{\mu,N}^{(PCH)}(x)}\right] \\ &= \mathtt{E}_{N,\alpha}^{(\mathrm{bc})}\left[\exp\left\{\beta\mu N_{\Lambda_N}^{(\ell)} - \frac{\beta a}{2|\Lambda_N|}\left(N_{\Lambda_N}^{(\ell)}\right)^2 + \frac{\beta b}{2|\Lambda_N|}\sum_{k\in A_N} k^2 \mathcal{N}_{\Lambda_N,k}^2\right\}\right]. \end{aligned} \tag{2.3.34}$$

Lemma 2.3.10. *Let $M \in \{PM, FCH, PCH\}$, $\alpha \leq 0$ and $\mu \in \mathbb{R}$. Then*

$$\nu_{N,\mu,\alpha}^{(M)} = \nu_{N,\mu+\alpha,0}^{(M)}. \tag{2.3.35}$$

PROOF. This follows from the linearity of these models' Hamiltonians in μ and the explicit expression of the ideal Bose gas model (2.3.5).

□

Remark 2.3.11. *Lemma 2.3.10 formalises the equivalence between the 'background'* *chemical potential and the 'energetic' chemical potential. This means that we are able to simplify our parameter space, and often we will - without loss of generality - set $\alpha = 0$ in our calculations. One notable exception to this will be in Chapter 6. Here the $\alpha = 0$ ideal gas will exist but the random variables we will be considering (particle numbers rather than anchor numbers) will not have exponential tightness. This will require us to set $\alpha < 0$ in our calculations. We will also adjust the energetic chemical potential then to be $\mu - \alpha$, so μ will then denote the total chemical potential.*

It is also worth noting how the GPM model fits in this α-μ equivalence. If we want to consider an $\alpha < 0$, we take $G(x) \mapsto -\alpha x + G(x)$. By the arguments of Lemma 2.3.10, this linear term can then be smuggled into the background measure and produce the same GPM measure. ◇

2.3.5 Condensation

The concept of "generalised condensation" was introduced in [Gir60], and the associated condensate density is derived for the true momentum Huang-Yang-Luttinger interaction model by [BLP88] with techniques previously used in [BLS84]. Inspired by this interpretation of condensation, we shall consider the following thermodynamic function.

Definition 2.3.12. For some interaction energy, H, suppose that the following limit exists:

$$\Delta^{(H)}(\beta, \mu) = \lim_{K\to\infty} \lim_{N\to\infty} \mathbb{E}_{\nu_N^H}\left[\sum_{j>K} j \boldsymbol{\lambda}_j^{(\Lambda_N)}\right]. \tag{2.3.36}$$

Then we call it the *condensate density.*

In the definition of the condensate density given in [Gir60], instead of considering particles occupying loops longer than some cut-off, we consider particles occupying momentum eigenstates with eigenvalue below some energy cut-off. Then we take the thermodynamic limit and take this cut-off to zero. A precise definition will be presented at equation (6.4.43) in Section 6.4 when required. Our definition heuristically follows from his by associating long loops with low energy states.

One aspect of this definition that may feel somewhat unsatisfactory is how we take the cut-off limit *after* the thermodynamic limit. In particular, we lose information on how the particles congregate in the relevant states as we take the limit. In Chapter 6, we will find a method that allows us to easily find a slightly different object. Our approach will be perfectly suited to evaluate

$$\Delta_A^{(H)}(\beta,\mu) = \lim_{N\to\infty} \mathbb{E}_{\nu_N^H}\left[\sum_{j\in A_N} j\boldsymbol{\lambda}_j^{(\Lambda_N)}\right], \tag{2.3.37}$$

where A_N is a sequence of subsets of $\mathbb{N}$ such that $\min A_N \to \infty$. Whilst the simultaneous cut-off and thermodynamic limit is appealing, it is usually a more difficult object to analyse. It is particular property of the interaction energy that will allow us to analyse it for that case.

In Chapter 5 and Section 6.4 we refer to a proof technique used by [BLP88] to derive the condensate density that used Griffith's Lemma. We repeat it here.

Lemma 2.3.13 (Griffith's Lemma). *Let $\{g_n(x)\}$ be a sequence of convex functions on the interval $I = (a,b) \subset \mathbb{R}$ with a point-wise limit $g(x)$, which is convex. Let $G_n^+(x)$ be the right derivatives of $g_n(x)$, and similarly for G^+. Let a '$-$' denote the left derivatives.*

Then for all $x \in I$,

$$\limsup_{n\to\infty} G_n^+(x) \le G^+(x) \tag{2.3.38}$$

$$\liminf_{n\to\infty} G_n^-(x) \ge G^-(x) \tag{2.3.39}$$

In particular, if all the $g_n(x)$ and $g(x)$ are differentiable at some point $x \in I$, then

$$\lim_{n\to\infty} \frac{\mathrm{d}g_n}{\mathrm{d}x}(x) = \frac{\mathrm{d}g}{\mathrm{d}x}(x). \tag{2.3.40}$$

Proof. The proof from [HL73] is short, so we repeat it here.

Fix $x \in I$. For $y > 0$ and $x \pm y \in I$,

$$g_n(x+y) \ge g_n(x) + yG_n^+(x), \qquad g_n(x-y) \ge g_n(x) - yG_n^-(x). \tag{2.3.41}$$

Fix y and take the limit $n \to \infty$. Then

$$\limsup_{n\to\infty} G_n^+ (x) \leq \frac{1}{y} (g (x + y) - g (x)), \tag{2.3.42}$$

and similarly for $\liminf_{n\to\infty} G_n^- (x)$. Now let $y \downarrow 0$.

$\square$

Chapter 3

Large Deviations for Interacting Stationary Empirical Measures

Recall from Chapter 2, the definition of the stationary empirical measure. A large deviation principle for this measure under zero interaction ($v \equiv 0$) was derived by Georgii and Zessin in [GZ93], and Adams, Collevecchio and König were able to introduce some non-zero interaction in some cases in [ACK11]. Specifically, they considered non-negative interactions ($v \geq 0$) for the canonical ensemble, and produced a large deviation principle when the prescribed physical particle density was below a critical density. Above this density they found upper and lower bounds on the thermodynamic free energy (an important step in deriving the large deviation principle), but these did not coincide. In particular, this LDP was not proven for densities higher than the non-interacting model's condensation critical density. Here we use the techniques of [ACK11] to derive a large deviation principle for the stationary empirical measure under the grand canonical ensemble with non-negative interactions. We are also able to cover the whole phase space by introducing the 'stabilising' term we introduced in Definition 2.2.8.

3.1 Positive Interactions

In this section we derive a level-3 large deviation principle for non-negative interactions in the grand-canonical ensemble with stabilisation for positive chemical potentials. This largely follows the approach taken by [ACK11], but the difficulties encountered by using the canonical ensemble are avoided and we include density-dependent stabilisation terms. Furthermore, we present some continuity lemmas that make many steps easier and provide greater insight into the overall result.

Before we present the main result, let us introduce some notation. Recall the Hamiltonian (H_Λ) given in (2.2.17), the stabilisation energy ($V_{\Lambda,\mu}$) given in (2.2.18), and the interacting measures ($\mathtt{Q}^{(H)}_{\Lambda,\mu,\alpha}$) given in (2.2.19). Let $U = \left[-\frac{1}{2}, \frac{1}{2}\right]^d \subset \mathbb{R}^d$ be the centred unit cube, and

$$\Phi\left(\omega\right) := \sum_{x\in\xi\cap U}\sum_{y\in\xi} T_{x,y}\left(\omega\right) \tag{3.1.1}$$

$$\mathcal{G}_\mu\left(x\right) := -\mu x + U_\mu\left(x\right) \tag{3.1.2}$$

$$W_\mu\left(P\right) := \mathcal{G}_\mu\left(\left\langle P, N_U^{(\ell)}\right\rangle\right). \tag{3.1.3}$$

In the following theorem we consider bc $= \varnothing$. For a discussion of how the bc $\in \{\mathrm{Dir}, \mathrm{per}\}$ cases can follow from this, see [ACK11, Section 3.4].

Theorem 3.1.1. *Suppose $v : \mathbb{R}_+ \to [0, +\infty]$, $\liminf_{r\to 0} v\left(r\right) > 0$, and that there exists $h > d$ such that $v\left(r\right) = O\left(r^{-h}\right)$ as $r \to \infty$. Then for $\alpha \leq 0$, $\mu \in \mathbb{R}$, the measures $\mathtt{Q}^{(H)}_{\Lambda_L,\mu,\alpha} \circ \left(\mathfrak{R}_{\Lambda_L,\cdot}\right)^{-1}$ satisfy, as $L \to \infty$, a large deviation principle in the topology $\tau_{\mathcal{L}}$ with rate $|\Lambda_L|$ and rate function*

$$I^{(v)}\left(P\right) = I_\alpha\left(P\right) + \beta W_\mu\left(P\right) + \left\langle P, \Phi\right\rangle - \inf_{Q\in\mathcal{P}_\theta}\left\{I_\alpha\left(Q\right) + \beta W_\mu\left(Q\right) + \left\langle Q, \Phi\right\rangle\right\}. \tag{3.1.4}$$

Remark 3.1.2. *It is worth noting that that the class of potentials, v, that this theorem can deal with includes hard-sphere potentials.* ◇

Corollary 3.1.3. *Suppose the hypotheses of Theorem 3.1.1 hold. Then for $\alpha \leq 0$ and $\mu \in \mathbb{R}$, the stabilised thermodynamic pressure*

$$p_U^{(v)}\left(\beta, \alpha, \mu\right) = \frac{1}{\beta}\overline{q}^{(\alpha)} + \lim_{L\to\infty}\frac{1}{\beta|\Lambda_L|}\mathtt{E}_\alpha\left[\mathrm{e}^{\left(\beta\mu N_\Lambda^{(\ell)} - \beta V_{\Lambda,\mu} - H_\Lambda\right)\left(\omega_{\mathrm{P}}\right)}\right] \tag{3.1.5}$$

$$= p\left(\beta, \alpha\right) + \frac{1}{\beta}\inf_{Q\in\mathcal{P}_\theta}\left\{I_\alpha\left(Q\right) + \beta W_\mu\left(Q\right) + \left\langle Q, \Phi\right\rangle\right\}. \tag{3.1.6}$$

If $\mathcal{G}_\mu \equiv 0$ then our model is 'not stabilised,' and for $\alpha \leq 0$

$$p^{(v)}(\beta,\alpha) = p(\beta,\alpha) + \frac{1}{\beta} \inf_{Q \in \mathcal{P}_\theta} \left\{ I_\alpha(Q) + \langle Q, \Phi \rangle \right\}. \tag{3.1.7}$$

Proof of Corollary 3.1.3. The first equality follows from Lemma 2.2.9. In the case of no interaction the log-expectation term vanishes, so the $\frac{1}{\beta}\overline{q}^{(\alpha)}$ term gives the non-interacting pressure, denoted $p(\beta,\alpha)$. In the interacting case, the log-expectation term is non-trivial but given by the constant term in the rate function given in Theorem 3.1.1. The non-stabilised model pressure is given by setting the stabilisation term to 0 and removing the μ parameter.

$\square$

Both the large deviation principle of [ACK11] and ours will be produced by building upon the base principle of [GZ93] - which used the topology of local convergence, $\tau_{\mathcal{L}}$, on $\mathcal{P}_\theta$. Recall from Definition 2.2.5 that this topology is the smallest such that the $\mathcal{P}_\theta \to [-\infty,+\infty]$ maps $P \mapsto \langle P, g\rangle$ are continuous for all local and tame g. This definition has the advantage of immediately giving us some continuity properties for some functions, and this is used in both [ACK11] and our following proof. It does not immediately give semicontinuity properties for such functions, nor continuity properties for maps onto $\mathcal{P}_\theta$, for example. The following two lemmas address some of these gaps.

Lemma 3.1.4. *Let $\Psi : \Omega \to [-\infty,+\infty]$ be local and satisfy*

$$\Psi(\omega) \geq -c'\left(1 + N_\Lambda(\omega)\right), \tag{3.1.8}$$

for some $c' > 0$ and finite box Λ. Then $P \mapsto \langle P, \Psi\rangle$ is lower $\tau_{\mathcal{L}}$-semicontinuous.

Remark 3.1.5. *Lemma 3.1.4 implies that if $\Psi : \Omega \to [0,+\infty]$ is local, then the map $P \mapsto \langle P, \Psi\rangle$ is lower $\tau_{\mathcal{L}}$-semicontinuous. Furthermore, the lemma tells us that if Ψ is local and satisfies $\Psi(\omega) \leq c'\left(1 + N_\Lambda(\omega)\right)$ for some $c' > 0$ and finite box Λ, then the map $P \mapsto \langle P, \Psi\rangle$ is upper $\tau_{\mathcal{L}}$-semicontinuous.* $\diamond$

Proof of Lemma 3.1.4. To prove that $P \mapsto \langle P, \Psi\rangle$ is lower $\tau_{\mathcal{L}}$-semicontinuous, we need to show that for all $a \in \mathbb{R}$,

$$\{P \in \mathcal{P}_\theta : \langle P, \Psi \rangle > a\} \in \tau_{\mathcal{L}}. \tag{3.1.9}$$

For all $c > 0$, define the functions

$$\Psi_c(\omega) = \min\{\Psi(\omega), c(1 + N_\Lambda(\omega))\}. \tag{3.1.10}$$

Since Ψ is local and satisfies (3.1.8), Ψ_c are local and tame. Hence the maps $P \mapsto \langle P, \Psi_c \rangle$ are all $\tau_{\mathcal{L}}$-continuous, and

$$\{P \in \mathcal{P}_\theta : \langle P, \Psi_c \rangle > a\} \in \tau_{\mathcal{L}}, \tag{3.1.11}$$

for all $a \in \mathbb{R}$ and $c > 0$. Furthermore, because $\Psi_c(\omega)$ are non-decreasing in c for a given ω, we have the increasing sequence of sets

$$\begin{aligned} c_1 \le c_2 \implies \{P \in \mathcal{P}_\theta : \langle P, \Psi_{c_1} \rangle > a\} &\subset \{P \in \mathcal{P}_\theta : \langle P, \Psi_{c_2} \rangle > a\} \\ &\subset \{P \in \mathcal{P}_\theta : \langle P, \Psi \rangle > a\}. \end{aligned} \tag{3.1.12}$$

We also note that $\Psi_c \to \Psi$ pointwise as $c \to \infty$.

We now make use of the decompositions

$$\Psi = (\Psi)_+ + (\Psi)_-, \qquad \Psi_c = (\Psi_c)_+ + (\Psi_c)_-. \tag{3.1.13}$$

Note that because $c(1 + N_\Lambda) > 0$, we have $(\Psi)_- = (\Psi_c)_-$ for all $c > 0$. Now fix $P \in \mathcal{P}_\theta$. Then by Fatou's Lemma,

$$\liminf_{c\to\infty} \left\langle P, (\Psi_c)_+ \right\rangle \ge \left\langle P, (\Psi)_+ \right\rangle. \tag{3.1.14}$$

Then because $\langle P, \Psi \rangle \ge \langle P, \Psi_c \rangle$,

$$\lim_{c\to\infty} \langle P, \Psi_c \rangle = \langle P, \Psi \rangle. \tag{3.1.15}$$

The combination of (3.1.12) and (3.1.15) means that

$$\{P \in \mathcal{P}_\theta : \langle P, \Psi \rangle > a\} = \bigcup_{c>0} \{P \in \mathcal{P}_\theta : \langle P, \Psi_c \rangle > a\} \in \tau_{\mathcal{L}}. \tag{3.1.16}$$

□

Definition 3.1.6. A map $p : \Omega \to \Omega$ is said to be a *thinning map* if

$$\xi \circ p(\omega) \subset \xi(\omega) \qquad \forall \omega \in \Omega. \tag{3.1.17}$$

In particular, this means that

$$N_\Lambda \circ p(\omega) \leq N_\Lambda(\omega), \qquad \forall \text{ open bounded cubes } \Lambda \subset \mathbb{R}^d, \omega \in \Omega. \tag{3.1.18}$$

Lemma 3.1.7. *Let the map $\pi : \mathcal{P}_\theta \to \mathcal{P}_\theta$ be that induced by a thinning map p. That is,*

$$\pi P = P \circ p^{-1}. \tag{3.1.19}$$

Then π is $\tau_{\mathcal{L}}$-continuous.

PROOF. Recall that $\tau_{\mathcal{L}}$ is the smallest topology on $\mathcal{P}_\theta$ such that for all local and tame functions F, the functions $P \mapsto \langle P, F \rangle$ are continuous. This means that all sets of the form

$$V = \{P \in \mathcal{P}_\theta : \langle P, F \rangle \in U\}, \tag{3.1.20}$$

where $F : \Omega \to \mathbb{R}$ is a local and tame function and $U \subset \mathbb{R}$ is open, are themselves open. Also, $\tau_{\mathcal{L}}$ is the smallest such topology, so these sets generate the whole topology. Since the preimage of an intersection (union) is the intersection (union) of the preimages, we only need concern ourselves with these sets to prove continuity of π.

We will show continuity by showing that the preimage of a set of the form (3.1.20) is also of this form. Using the definition of the preimage and (3.1.20) we have

$$\pi^{-1}V = \{P \in \mathcal{P}_\theta : \pi P \in V\} \tag{3.1.21}$$

$$= \{P \in \mathcal{P}_\theta : \langle \pi P, F\rangle \in U\} \tag{3.1.22}$$

$$= \{P \in \mathcal{P}_\theta : \langle P, F \circ p\rangle \in U\}. \tag{3.1.23}$$

In this last line we have performed the calculation:

$$\langle \pi P, F\rangle = \int F \mathrm{d}\left(\pi P\right) = \int F \mathrm{d}\left(P \circ p^{-1}\right) = \int F \circ p \mathrm{d}P = \langle P, F \circ p\rangle. \tag{3.1.24}$$

Since F is tame and p is a thinning, $|F \circ p| \leq c\left(1 + N_\Lambda \circ p\right) \leq c\left(1 + N_\Lambda\right)$. Hence $F \circ p$ is also tame. Locality also follows from the locality of F. We thus have $\pi^{-1}V$ in the desired form, and the continuity of π.

□

Remark 3.1.8. *In particular, Lemma 3.1.7 can be applied with p being a projection. This, with the Contraction Principle (Lemma 2.1.15), allows for an alternative way to derive a large deviation principle for the hard-sphere potential: we remove any anchors with particles that encroach on some other particle.* ◇

Proving the large deviation principle comes down to proving the existence on the thermodynamic pressure. This was discussed in the proof of Theorem 2.1.5. Up to a simple ideal gas term, finding the pressure is equivalent to proving that the limit

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{E}_\alpha \left[\mathrm{e}^{\beta\mu N^{(\ell)}_{\Lambda_L} - \beta V_{\Lambda_L,\mu} - H_{\Lambda_L}}\right] = -\inf_{P\in\mathcal{P}_\theta} \left\{I_\alpha\left(P\right) + \beta W_\mu\left(P\right) + \langle P, \Phi\rangle\right\} \tag{3.1.25}$$

holds.

3.1.1 Pressure Upper Bound

Lemma 3.1.9. *Suppose $v(r) \geq 0$ for all $r \geq 0$. Then*

$$\limsup_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{E}_\alpha \left[\mathrm{e}^{\beta\mu N^{(\ell)}_{\Lambda_L} - \beta V_{\Lambda_L,\mu} - H_{\Lambda_L}} \right] \leq - \inf_{P\in\mathcal{P}_\theta} \left\{ I_\alpha (P) + \beta W_\mu (P) + \langle P, \Phi \rangle \right\}. \tag{3.1.26}$$

Our strategy here is to approximate the Hamiltonian H_Λ by integrating some local and tame test function ($\Phi^{(R,M,S)}$) with respect to the stationary empirical field. We will then apply large deviation techniques for this tilt and finally relax the approximation parameters. Let us define

$$\Phi^{(R,M,S)} (\omega) = \sum_{x\in\xi\cap U} \sum_{y\in\xi} T^{(R,M)}_{x,y} (\omega) \mathbb{1} \left\{ N_{\Lambda_R} (\omega) \leq S \right\} \tag{3.1.27}$$

$$T^{(R,M)}_{x,y} (\omega) = \begin{cases} \min \left\{ T_{x,y} (\omega), M \right\} & : |x-y|_\infty \leq R \\ 0 & : |x-y|_\infty > R, \end{cases} \tag{3.1.28}$$

$$N^{(\ell,K)}_U (\omega) = \sum_{k=1}^{K} k \mathcal{N}_{U,k} (\omega), \tag{3.1.29}$$

$$W^{(K)}_\mu (P) = \mathcal{G}_\mu \left(\left\langle P, N^{(\ell,K)}_U \right\rangle \right), \tag{3.1.30}$$

for $R, M, S \in [0, +\infty]$, $K \in \mathbb{N}$, $\omega \in \Omega$ and $P \in \mathcal{P}_\theta$.

Lemma 3.1.10. *Fix $\Lambda = \Lambda_L$. Then for $R, M, S \in (3, +\infty)$ and $L \geq R + 1$,*

$$H_\Lambda (\omega) \geq |\Lambda| \langle \mathfrak{R}_{\Lambda,\omega}, \Phi^{(R,M,S)} \rangle - 6^d M S N_{\Lambda_L \setminus \Lambda_{L-R-1}} (\omega). \tag{3.1.31}$$

Proof. This follows the general strategy of [ACK11, Lemma 3.2 i)], adapted to our cut-off parameters. Let us begin by evaluating

$$|\Lambda| \langle \mathfrak{R}_{\Lambda,\omega}, \Phi^{(R,M,S)} \rangle = \sum_{x,y\in\xi_{(\Lambda)}} T^{(R,M)}_{x,y} (\omega_{(\Lambda)}) \int_{\Lambda\cap(U-x)} \mathrm{d}z \mathbb{1} \left\{ \# \left(\xi_{(\Lambda)} \cap (\Lambda_R - z) \right) \leq S \right\}. \tag{3.1.32}$$

Now we split off the $(x,y) \in \Lambda^2$ terms from the sum. On these we bound

$$\int_{\Lambda\cap(U-x)} \mathrm{d}z \mathbb{1}\left\{\#\left(\xi_{(\Lambda)} \cap \left(\Lambda_R - z\right)\right) \leq S\right\} \leq 1 \tag{3.1.33}$$

$$T^{(R,M)}_{x,y}\left(\omega_{(\Lambda)}\right) = T^{(R,M)}_{x,y}\left(\omega\right) \leq T_{x,y}\left(\omega\right). \tag{3.1.34}$$

Therefore we have

$$\left|\Lambda\right| \left\langle \mathfrak{R}_{\Lambda,\omega}, \Phi^{(R,M,S)}\right\rangle \leq H_\Lambda\left(\omega\right) + \Psi^{(R,M,S)}_\Lambda\left(\omega\right), \tag{3.1.35}$$

where

$$\Psi^{(R,M,S)}_\Lambda\left(\omega\right) = \sum_{x,y\in\xi_{(\Lambda)}:(x,y)\notin\Lambda^2} T^{(R,M)}_{x,y}\left(\omega_{(\Lambda)}\right) \int_{\Lambda\cap(U-x)} \mathrm{d}z \mathbb{1}\left\{\#\left(\xi_{(\Lambda)} \cap \left(\Lambda_R - z\right)\right) \leq S\right\} \tag{3.1.36}$$

$$\leq M \sum_{\substack{x,y\in\xi_{(\Lambda)}:\\ x-y\in\Lambda_R,(x,y)\notin\Lambda^2}} \mathbb{1}\left\{\exists z \in \Lambda\cap\left(U-x\right) : \#\left(\xi_{(\Lambda)} \cap \left(\Lambda_R - z\right)\right) \leq S\right\} \tag{3.1.37}$$

$$\leq M \sum_{\substack{x,y\in\xi_{(\Lambda)}:\\ x\in\Lambda_{L+1},x-y\in\Lambda_R,(x,y)\notin\Lambda^2}} \mathbb{1}\left\{\#\left(\xi_{(\Lambda)} \cap \left(\Lambda_{R-1} + x\right)\right) \leq S\right\}. \tag{3.1.38}$$

In this last inequality, the $x \in \Lambda_{L+1}$ condition appears in the sum because otherwise we would have $\Lambda\cap(U-x) = \varnothing$. Then given any $z \in \Lambda\cap(U-x)$, the box $\Lambda_{R-1}+x$ is a subset of the box $\Lambda_R - z$. Therefore the indicator function can be bounded as done here.

Now we split the sum into

1. $x \in \Lambda_{L+1} \setminus \Lambda_L$, and $y \in \Lambda_R + x \subset \Lambda_{L+R+1} \setminus \Lambda_{L-R}$,

2. $x \in \Lambda_{L+1}$, and $y \in (\Lambda_R + x) \setminus \Lambda_L \subset \Lambda_{L+R+1} \setminus \Lambda_{L-R}$.

Since, in both cases, the index y is summed over some subset of $\Lambda_{L+R+1} \setminus \Lambda_{L-R}$, we have

$$\Psi^{(R,M,S)}_{\Lambda}(\omega) \le M \sum_{y\in\xi_{(\Lambda)}\cap(\Lambda_{L+R+1}\setminus\Lambda_{L-R})} \#\left\{x\in\xi_{(\Lambda)}\cap(\Lambda_R+y) : \#\left(\xi_{(\Lambda)}\cap(\Lambda_{R-1}+x)\right)\le S\right\}. \tag{3.1.39}$$

To bound the number of such x, we cover Λ_R+y with r boxes $\Delta_1,\ldots,\Delta_r$ of diameter $\frac{1}{2}(R-1)$. We can choose these so that $r=\lceil 2R/(R-1)\rceil^d$, and therefore $r=3^d$ for $R>3$. Then because $\Delta_i\subset\Lambda_{R-1}+x$ if $x\in\Delta_i$,

$$\begin{aligned}
&\#\left\{x\in\xi_{(\Lambda)}\cap(\Lambda_R+y) : \#\left(\xi_{(\Lambda)}\cap(\Lambda_{R-1}+x)\right)\le S\right\}\\
&\qquad\le\sum_{i=1}^{r}\#\left\{x\in\xi_{(\Lambda)}\cap\Delta_i : \#\left(\xi_{(\Lambda)}\cap(\Lambda_{R-1}+x)\right)\le S\right\} &(3.1.40)\\
&\qquad\le\sum_{i=1}^{r}\#\left\{x\in\xi_{(\Lambda)}\cap\Delta_i : \#\left(\xi_{(\Lambda)}\cap\Delta_i\right)\le S\right\} &(3.1.41)\\
&\qquad\le rS=3^dS. &(3.1.42)
\end{aligned}$$

Therefore $\Psi^{(R,M,S)}_{\Lambda}(\omega)\le 3^dMSN_{\Lambda_{L+R+1}\setminus\Lambda_{L-R}}(\omega_{(\Lambda)})$. Finally, from the periodicity of $\omega_{(\Lambda)}$, we have

$$N_{\Lambda_{L+R+1}\setminus\Lambda_{L-R}}(\omega_{(\Lambda)})\le 2^dN_{\Lambda\setminus\Lambda_{L-R-1}}(\omega). \tag{3.1.43}$$

This provides the required bound.

□

For the stabilisation term, we approximate by only considering cycles of length less than some cut-off. First we recall [ACK11, Lemma 3.1].

Lemma 3.1.11. *For any $\omega\in\Omega$,*

$$N^{(\ell)}_{\Lambda_L}(\omega)=|\Lambda_L|\left\langle\mathfrak{R}_{\Lambda_L,\omega},N^{(\ell)}_U\right\rangle. \tag{3.1.44}$$

Then,

$$V_{\Lambda_L,\mu}(\omega)-\mu N^{(\ell)}_{\Lambda_L}(\omega)=|\Lambda_L|W_\mu(\mathfrak{R}_{\Lambda_L,\omega})\ge|\Lambda_L|W^{(K)}_\mu(\mathfrak{R}_{\Lambda_L,\omega}). \tag{3.1.45}$$

PROOF. The first part follows from considering Palm measures and [GZ93, Remark 2.3(1)]. Here we summarise the direct proof from [ACK11]. From explicit calculation,

$$|\Lambda_L| \left\langle \mathfrak{R}_{\Lambda_L,\omega}, N_U^{(\ell)} \right\rangle = N_{\Lambda_L}^{(\ell)}(\omega) + \sum_{x \in \xi_{(\Lambda_L)} \cap ((\Lambda_L + U)\setminus\Lambda_L)} \ell(f_x) |\Lambda_L \cap (U - x)| \\ + \sum_{x \in \xi \cap \Lambda_L} \ell(f_x) (|\Lambda_L \cap (U - x)| - 1). \quad (3.1.46)$$

Then by considering the copies of a given $x \in \Lambda_L$ resulting from the periodic $\xi_{(\Lambda_L)}$ in adjacent boxes, it can be shown that the last two sums perfectly cancel.

For the first part of (3.1.45), we can then substitute (3.1.44) into the expression (2.2.18). Note that G_μ is non-decreasing and $N_U^{(\ell,K)}$ is non-decreasing in K to get $W_\mu(\mathfrak{R}_{\Lambda_L,\omega}) \geq W_\mu^{(K)}(\mathfrak{R}_{\Lambda_L,\omega})$.

□

Lemma 3.1.12. *The functions $P \mapsto W_\mu^{(K)}(P)$ and $P \mapsto \langle P, \Phi^{(R,M,S)} \rangle$ are $\tau_{\mathcal{L}}$-continuous. The function $P \mapsto W_\mu(P)$ is lower $\tau_{\mathcal{L}}$-semicontinuous.*

PROOF. Note that $P \mapsto \left\langle P, N_U^{(\ell,K)} \right\rangle$ is continuous because $N_U^{(\ell,K)}$ is local and tame ($N_U^{(\ell,K)} \leq K N_U$). Then recall that $\mathcal{G}_\mu$ is continuous, so the composition $W_\mu^{(K)}$ is continuous.

The function $\Phi^{(R,M,S)} : \Omega \to [0, +\infty]$ is clearly local, and can be bounded with

$$0 \leq \Phi^{(R,M,S)}(\omega) \leq \sum_{x \in \xi \cap \Lambda} \sum_{y \in \xi \cap \Lambda_{R+1}} M \mathbb{1}\{N_{\Lambda_R}(\omega) \leq S\} \leq MSN_{\Lambda_{R+1}}(\omega). \quad (3.1.47)$$

Therefore $\Phi^{(R,M,S)}$ is tame, and the map $P \mapsto \langle P, \Phi^{(R,M,S)} \rangle$ is $\tau_{\mathcal{L}}$-continuous.

From Lemma 3.1.4, we know that $P \mapsto \left\langle P, N_U^{(\ell)} \right\rangle$ is lower semicontinuous. Then since $\mathcal{G}_\mu$ is continuous and non-decreasing we have lower semicontinuity of the composition W_μ from Lemma 4.3.2.

□

By a Hölder-type inequality, Lemmas 3.1.10 and 3.1.11 gives us

$$\begin{aligned}\mathtt{E}_\alpha &\left[\mathrm{e}^{\beta\mu N_\Lambda^{(\ell)}-\beta V_{\Lambda,\mu}-H_\Lambda}\right]\\ &\leq \mathtt{E}_\alpha\left[\exp\left(-\frac{1}{1-\eta}|\Lambda|\left(\beta W_\mu^{(K)}\left(\mathfrak{R}_{\Lambda,\cdot}\right)+\langle\mathfrak{R}_{\Lambda,\cdot},\Phi^{(R,M,S)}\rangle\right)\right)\right]^{1-\eta}\\ &\qquad\qquad\times\mathtt{E}_\alpha\left[\exp\left(\frac{1}{\eta}6^d MSN_{\Lambda_L\setminus\Lambda_{L-R-1}}\right)\right]^{\eta},\end{aligned}\tag{3.1.48}$$

for $\eta\in(0,1)$. Now, under the reference measure, $N_{\Lambda_L\setminus\Lambda_{L-R-1}}$ is a Poisson random variable with mean $\overline{q}^{(\alpha)}|\Lambda_L\setminus\Lambda_{L-R-1}|=O\left(L^{d-1}\right)$. Therefore

$$\begin{aligned}\mathtt{E}_\alpha\left[\exp\left(\frac{1}{\eta}6^d MSN_{\Lambda_L\setminus\Lambda_{L-R-1}}\right)\right]^{\eta}&=\exp\left(\eta\overline{q}^{(\alpha)}|\Lambda_L\setminus\Lambda_{L-R-1}|\left(\mathrm{e}^{\frac{1}{\eta}6^d MS}-1\right)\right)\\ &=\mathrm{e}^{o(|\Lambda|)},\end{aligned}\tag{3.1.49}$$

so we can neglect this factor.

Lemma 3.1.13.

$$\limsup_{L\to\infty}\frac{1}{|\Lambda|}\log\mathtt{E}_\alpha\left[\mathrm{e}^{\beta\mu N_\Lambda^{(\ell)}-\beta V_{\Lambda,\mu}-H_\Lambda}\right]\leq-\inf_{P\in\mathcal{P}_\theta}\left\{I_\alpha\left(P\right)+\beta W_\mu\left(P\right)+\langle P,\Phi\rangle\right\}\tag{3.1.50}$$

Proof. Let us define $F:\mathcal{P}_\theta\to[0,+\infty]$ and $F_{R,M,S,K,\eta}:\mathcal{P}_\theta\to[0,+\infty]$ by

$$F:P\mapsto\beta W_\mu\left(P\right)+\langle P,\Phi\rangle\tag{3.1.51}$$

$$F_{R,M,S,K,\eta}:P\mapsto\frac{1}{1-\eta}\left(\beta W_\mu^{(K)}\left(P\right)+\langle P,\Phi^{(R,M,S)}\rangle\right).\tag{3.1.52}$$

From Lemma 3.1.12, $F_{R,M,S,K,\eta}$ is $\tau_{\mathcal{L}}$-continuous. Note that $F_{R,\infty,\infty,\infty,0}$ is lower $\tau_{\mathcal{L}}$-semicontinuous from the lower semicontinuity of W_μ (see Lemma 3.1.12) and $\Phi^{(R,\infty,\infty)}$ being local and non-negative (see Lemma 3.1.4). Since $F_{R,M,S,K,\eta}$ is bounded below, we can apply Varadhan's Lemma (Theorem 2.1.5) with Theorem 2.2.6 to get

$$\lim_{L\to\infty}\frac{1}{|\Lambda|}\log\mathtt{E}_\alpha\left[\exp\left(-|\Lambda|F_{R,M,S,K,\eta}\left(\mathfrak{R}_{\Lambda,\cdot}\right)\right)\right]=-\inf_{P\in\mathcal{P}_\theta}\left\{I_\alpha\left(P\right)+F_{R,M,S,K,\eta}\left(P\right)\right\}.\tag{3.1.53}$$

To prove our result, we now only need to show that

$$\liminf_{R\to\infty} \liminf_{M,S,K\to\infty,\eta\downarrow 0} \inf_{P\in\mathcal{P}_\theta} \left\{I_\alpha\left(P\right)+F_{R,M,S,K,\eta}\left(P\right)\right\} \geq \inf_{P\in\mathcal{P}_\theta} \left\{I_\alpha\left(P\right)+F\left(P\right)\right\}. \tag{3.1.54}$$

Recall that W_μ and $W_\mu^{(K)}$ are all bounded below by $\inf \mathcal{G}_\mu$. Therefore we can add a constant that ensures that $W_\mu, W_\mu^{(K)} \geq 0$, and remove it at the end of the calculation. Therefore we proceed without loss of generality with $W_\mu, W_\mu^{(K)} \geq 0$.

Fix R. Pick $M_n, S_n, K_n \to \infty$, $\eta_n \downarrow 0$, and some $Q_n \in \mathcal{P}_\theta$ such that

$$I_\alpha\left(Q_n\right)+F_{R,M_n,S_n,K_n,\eta_n}\left(Q_n\right) < \inf_{P\in\mathcal{P}_\theta} \left\{I_\alpha\left(P\right)+F_{R,M_n,S_n,K_n,\eta_n}\left(P\right)\right\}+\frac{1}{n}. \tag{3.1.55}$$

Since I has compact level sets, and (without loss of generality $F_{R,M,S,K,\eta} \geq 0$), we may assume $Q=\lim_{n\to\infty} Q_n$ exists in $\mathcal{P}_\theta$. Now fix any large M, S and K, and small η. Then for sufficiently large n, $F_{R,M_n,S_n,K_n\eta_n}\left(P\right) \geq F_{R,M,S,K,\eta}\left(P\right)$ for all P and

$$\inf_{P\in\mathcal{P}_\theta} \left\{I_\alpha\left(P\right)+F_{R,M_n,S_n,K_n,\eta_n}\left(P\right)\right\} > I_\alpha\left(Q_n\right)+F_{R,M,S,K,\eta}\left(Q_n\right)-\frac{1}{n}. \tag{3.1.56}$$

Now we can send $n\to\infty$ and use the lower semicontinuity of I_α and $F_{R,M,S,K,\eta}$ to get

$$\liminf_{n\to\infty} \inf_{P\in\mathcal{P}_\theta} \left\{I_\alpha\left(P\right)+F_{R,M_n,S_n,K_n,\eta_n}\left(P\right)\right\} \geq I\left(Q\right)+F_{R,M,S,K,\eta}\left(Q\right). \tag{3.1.57}$$

Then the monotone convergence theorem gives

$$\liminf_{M,S\to\infty,\eta\downarrow 0} \inf_{P\in\mathcal{P}_\theta} \left\{I_\alpha\left(P\right)+F_{R,M,S,K,\eta}\left(P\right)\right\} \geq \inf_{P\in\mathcal{P}_\theta} \left\{I\left(P\right)+F_{R,\infty,\infty,\infty,0}\left(P\right)\right\}. \tag{3.1.58}$$

Since $F_{R,\infty,\infty,\infty,0}$ is still lower semicontinuous, we can repeat this argument to take $R\to\infty$, and (3.1.54) is proven as required.

□

3.1.2 Pressure Lower Bound

Lemma 3.1.14. *Suppose $v(r) \geq 0$ for all $r \geq 0$, $\liminf_{r\to 0} v(r) > 0$, and that there exists $h > d$ such that $v(r) = O\left(r^{-h}\right)$ as $r \to \infty$. Then*

$$\liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{E}_\alpha \left[\mathrm{e}^{\beta\mu N^{(\ell)}_{\Lambda_L} - \beta V_{\Lambda_L,\mu} - H_{\Lambda_L}} \right] \geq - \inf_{P\in\mathcal{P}_\theta} \left\{ I_\alpha(P) + \beta W_\mu(P) + \langle P, \Phi\rangle \right\}. \tag{3.1.59}$$

Now instead of adjusting the tilt to use large deviation techniques, now we focus on restricting the probability space to make the tilt well behaved. We will restrict $\mathcal{P}_\theta$ to make the map $P \mapsto \beta W_\mu(P) + \langle P, \Phi\rangle$ continuous, and follow a standard strategy of changing the measure so that such untypical events become typical. An ergodic approximation is then carried out so that McMillan's Ergodic Theorem can be applied to control the Radon-Nikodym density of the transformed process. Proofs of the appropriate version of McMillan's Theorem can be found in [Fri70] and [NZ79]. Let us first relate H_Λ and Φ.

Lemma 3.1.15. *Fix $\Lambda = \Lambda_L$. Then*

$$H_\Lambda(\omega) \leq |\Lambda| \langle \mathfrak{R}_{\Lambda,\omega}, \Phi\rangle. \tag{3.1.60}$$

Proof. This is precisely the proof of [ACK11, Lemma 3.2 ii)]. The argument proceeds similarly to that of the first part of Lemma 3.1.11, but now the excess terms in (3.1.46) are sums over pairs of points. The second in the pair is not restricted to be within any given compact region, and so we have terms that will not be cancelled.

□

Now approximate the mark space E with the cut-off version

$$E^{(K,R)} = \bigcup_{k=1}^{K} \mathcal{C}_{k,R}, \qquad \text{where} \quad \mathcal{C}_{k,R} = \left\{ f \in \mathcal{C}_k : \sup_{s\in[0,k\beta]} |f(s) - f(0)| \leq R \right\}. \tag{3.1.61}$$

Let $\Omega^{(K,R)}$ denote the set of locally finite point measures on $\mathbb{R}^d \times E^{(K,R)}$, and define the projections

$$\pi_{K,R} : \Omega \to \Omega^{(K,R)} \tag{3.1.62}$$

$$\omega \mapsto \omega^{(K,R)} = \sum_{x \in \xi : f_x \in E^{(K,R)}} \delta_{(x,f_x)}. \tag{3.1.63}$$

On $\Omega^{(K,R)}$ we consider $\omega_P^{(K,R)} = \pi_{K,R}(\omega_P)$ as the reference process, with distribution denoted $\mathtt{Q}_\alpha^{(K,R)}$. Its intensity measure is $\nu^{(K,R)} = \sum_{k=1}^K \nu_k^{(K,R)}$, where $\nu_k^{(K,R)}$ is the restriction of ν_k to $\Omega^{(K,R)}$. By $I_\alpha^{(K,R)}$ we denote the rate function with respect to $\omega_P^{(K,R)}$ (that is, ω_P replaced with $\omega_P^{(K,R)}$). A variant of Theorem 2.2.6 gives that $\left\{\mathfrak{R}_{\Lambda_L,\omega_P^{(K,R)}}\right\}_{L\geq 0}$ satisfies the large deviation principle with rate function $I_\alpha^{(K,R)}$. Observe that $\mathfrak{R}_{\Lambda_L,\omega_P^{(K,R)}} = \mathfrak{R}_{\Lambda_L,\omega_P} \circ \pi_{K,R}^{-1}$, and that $P \mapsto P \circ \pi_{K,R}^{-1}$ is continuous (apply Lemma 3.1.7). Therefore the contraction principle gives

$$I_\alpha^{(K,R)}(P) = \inf\left\{I_\alpha(Q) : Q \in \mathcal{P}_\theta, Q \circ \pi_{K,R}^{-1} = P\right\}. \tag{3.1.64}$$

For the moment we keep the parameters K, R fixed. For brevity, write $\mathfrak{R}_{\Lambda_L,\omega_P} = \mathfrak{R}_L$ and $\mathfrak{R}_{\Lambda_L,\omega_P^{(K,R)}} = \mathfrak{R}_L^{(K,R)}$ Then we have the following bound on the expectation:

$$\mathtt{E}_\alpha\left[\mathrm{e}^{\beta\mu N_\Lambda^{(\ell)} - \beta V_{\Lambda,\mu} - H_\Lambda}\right] \geq \mathtt{E}_\alpha\left[\mathrm{e}^{-|\Lambda|(\beta W_\mu(\mathfrak{R}_L) + \langle \mathfrak{R}_L, \Phi\rangle)}\right] \tag{3.1.65}$$

$$\geq \mathtt{E}_\alpha\left[\mathrm{e}^{-|\Lambda|(\beta W_\mu(\mathfrak{R}_L) + \langle \mathfrak{R}_L, \Phi\rangle)} \mathbb{1}\left\{\mathfrak{R}_L \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)\right\}\right] \tag{3.1.66}$$

$$= \mathtt{E}_\alpha\left[\mathrm{e}^{-|\Lambda|\left(\beta W_\mu\left(\mathfrak{R}_L^{(K,R)}\right) + \left\langle \mathfrak{R}_L^{(K,R)}, \Phi\right\rangle\right)} \mathbb{1}\left\{\mathfrak{R}_L \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)\right\}\right] \tag{3.1.67}$$

$$= \mathtt{E}_\alpha^{(K,R)}\left[\mathrm{e}^{-|\Lambda|\left(\beta W_\mu\left(\mathfrak{R}_L^{(K,R)}\right) + \left\langle \mathfrak{R}_L^{(K,R)}, \Phi\right\rangle\right)}\right] \mathtt{Q}_\alpha\left(\mathfrak{R}_L \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)\right), \tag{3.1.68}$$

where in the last equality we have used the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathtt{Q}_\alpha^{(K,R)}}{\mathrm{d}\mathtt{Q}_\alpha}(A) = \frac{1}{\mathtt{Q}_\alpha\left(\mathfrak{R}_L \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)\right)} \mathbb{1}\left\{A \cap \left\{\mathfrak{R}_L \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)\right\}\right\}, \tag{3.1.69}$$

for measurable $A \subset \mathcal{P}_\theta$.

Lemma 3.1.16.

$$\lim_{K\to\infty}\lim_{R\to\infty}\liminf_{L\to\infty}\frac{1}{|\Lambda_L|}\log \mathsf{Q}_\alpha\left(\mathfrak{R}_L\in\mathcal{P}_\theta\left(\Omega^{(K,R)}\right)\right)=0. \tag{3.1.70}$$

Proof. Having the event $\{\mathfrak{R}_L\in\mathcal{P}_\theta\left(\Omega^{(K,R)}\right)\}$ is equivalent to having both the following two conditions on ω:

1): $\ell\left(f_x\right)\leq K$ for all $x\in\xi\cap\Lambda_L$,

2): $\sup_{s\in[0,\beta k]}|f_x\left(s\right)-x|\leq R$ for all $x\in\xi\cap\Lambda_L$ such that $\ell\left(f_x\right)=k$, for all k.

Let us first consider the probability of event *1*, namely $\mathsf{Q}_\alpha\left(\mathit{1}\right)$. Recall that under Q_α the functions $\mathcal{N}_{\Lambda,k}\left(\omega\right)=\#\left\{x\in\xi\cap\Lambda:\ell\left(f_x\right)=k\right\}$ are independent Poisson random variables with means $|\Lambda|q_k\mathrm{e}^{\beta\alpha k}$. Therefore

$$\mathsf{Q}_\alpha\left(\mathit{1}\right)=\exp\left(-|\Lambda_L|\sum_{k>K}q_k\mathrm{e}^{\beta\alpha k}\right). \tag{3.1.71}$$

Now we consider *2)*, conditioning upon *1)* and k. Denote

$$Q_{R,k}=\mathsf{Q}_\alpha\left(\sup_{s\in[0,\beta k]}|f_x\left(s\right)-x|\leq R\quad|\quad\left(x,f_x\right)\in\xi\times\mathcal{C}_k\right)\in\left(0,1\right). \tag{3.1.72}$$

Note that $Q_{R,k}$ is independent of x, and is increasing in R and decreasing in k. In particular, $Q_{R,k}\to 1$ as $R\to\infty$ for fixed k. Then using the independence of the marks, for any $\delta>0$,

$$\begin{aligned}\mathsf{Q}_\alpha\left(\mathit{2}|\mathit{1}\right)&=\prod_{k\leq K}\sum_{n\in\mathbb{N}}\left(Q_{R,k}\right)^n\mathsf{Q}_\alpha\left(\mathcal{N}_{\Lambda_L,k}=n\right) &(3.1.73)\\ &\geq\prod_{k\leq K}\sum_{n\leq|\Lambda_L|q_k\mathrm{e}^{\beta\alpha k}(1+\delta)}\left(Q_{R,k}\right)^n\mathsf{Q}_\alpha\left(\mathcal{N}_{\Lambda_L,k}=n\right) &(3.1.74)\\ &\geq\prod_{k\leq K}\left(Q_{R,k}\right)^{|\Lambda_L|q_k\mathrm{e}^{\beta\alpha k}(1+\delta)}\mathsf{Q}_\alpha\left(\mathcal{N}_{\Lambda_L,k}\leq|\Lambda_L|q_k\mathrm{e}^{\beta\alpha k}\left(1+\delta\right)\right). &(3.1.75)\end{aligned}$$

By the Chebyshev's inequality, $\mathsf{Q}_\alpha\left(\mathcal{N}_{\Lambda_L,k}\leq|\Lambda_L|q_k\mathrm{e}^{\beta\alpha k}\left(1+\delta\right)\right)\to 1$ as $L\to\infty$.

Hence

$$\liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{Q}_\alpha\left(\mathit{1} \cap \mathit{2}\right) \geq -\sum_{k>K} q_k \mathrm{e}^{\beta\alpha k} + (1+\delta) \sum_{k\leq K} q_k \mathrm{e}^{\beta\alpha k} \log Q_{R,k} \tag{3.1.76}$$

$$\geq -\sum_{k>K} q_k \mathrm{e}^{\beta\alpha k} + (1+\delta)\, \overline{q}^{(\alpha)} \log Q_{R,K} \tag{3.1.77}$$

$$\liminf_{R\to\infty} \liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{Q}_\alpha\left(\mathit{1} \cap \mathit{2}\right) \geq -\sum_{k>K} q_k \mathrm{e}^{\alpha k}. \tag{3.1.78}$$

Note that the event $\mathit{2}$ is monotone increasing in R and so the $R \to \infty$ limit exists. Finally, because $q \in \ell_1$ and $\alpha \leq 0$, taking $K \to \infty$ gives

$$\lim_{K\to\infty} \lim_{R\to\infty} \liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{Q}_\alpha\left(\mathit{1} \cap \mathit{2}\right) = 0. \tag{3.1.79}$$

□

As a result of Lemma 3.1.16, we only need to show

$$\liminf_{K\to\infty} \liminf_{R\to\infty} \liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{E}_\alpha^{(K,R)} \left[\mathrm{e}^{-|\Lambda|\left(\beta W_\mu\left(\mathfrak{R}_L^{(K,R)}\right) + \left\langle \mathfrak{R}_L^{(K,R)}, \Phi\right\rangle\right)}\right]$$
$$\geq -\inf_{P\in\mathcal{P}_\theta} \left\{I_\alpha\left(P\right) + \beta W_\mu\left(P\right) + \left\langle P, \Phi\right\rangle\right\}. \tag{3.1.80}$$

Lemma 3.1.17. *On $\mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$, the function $P \mapsto W_\mu\left(P\right)$ is continuous.*

Proof. On $\mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$, $0 \leq N_U^{(\ell)}\left(\omega\right) \leq K N_U\left(\omega\right)$, and $N_U^{(\ell)}$ is, of course, local. Hence $P \mapsto \left\langle P, N_U^{(\ell)}\right\rangle$ is continuous. W_μ is therefore a composition of continuous functions and is itself continuous.

□

Our main issue now is that the map $P \mapsto \left\langle P, \Phi\right\rangle$ is not necessarily upper semicontinuous on $\mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$. We can correct this with a further restriction. For

$r \in (0, \infty)$, define

$$\Gamma_r = \left\{ \omega \in \Omega^{(K,R)} : T_{x,y}(\omega) \leq r \quad \forall x, y \in \xi, \text{ and } |x - y| \geq \frac{1}{r} \,\forall \text{ distinct } x, y \in \xi \right\} \tag{3.1.81}$$

$$\mathcal{P}_{\theta,r}^{(K,R)} = \left\{ P \in \mathcal{P}_\theta \left(\Omega^{(K,R)} \right) : P(\Gamma_r) = 1 \right\}. \tag{3.1.82}$$

We now use the fact that $t \mapsto t^{d-1} \sup_{s \geq t - 2R} v(s)$ is integrable. This follows from having $h > d$ such that $v(r) = O\left(r^{-h}\right)$ as $r \to \infty$.

Lemma 3.1.18. *For all $r > 0$, the map $P \mapsto \langle P, \Phi \rangle$ is continuous on $\mathcal{P}_{\theta,r}^{(K,R)}$.*

Proof. Define $\pi_n : \Omega \to \Omega$ to be the projection

$$\pi_n(\omega) = \sum_{x \in \xi \cap [-n,n]^d} \delta_{(x, f_x)}. \tag{3.1.83}$$

Given $P \in \mathcal{P}_{\theta,r}^{(K,R)}$, let $P_n = P \circ \pi_n^{-1}$. Also let $\{P^{(\alpha)}\}_{\alpha \in D}$ be a net in $\mathcal{P}_{\theta,r}^{(K,R)}$ such that $P^{(\alpha)} \to P$. Then for any $n \in \mathbb{N}$ and $\alpha \in D$,

$$\begin{aligned}
|\langle P, \Phi \rangle - \langle P^{(\alpha)}, \Phi \rangle| &\leq |\langle P, \Phi - \Phi \circ \pi_n \rangle| + |\langle P^{(\alpha)} - P, \Phi \circ \pi_n \rangle| \\
&\qquad + \sup_{\alpha \in D} |\langle P^{(\alpha)}, \Phi - \Phi \circ \pi_n \rangle| &(3.1.84)\\
&\leq |\langle P^{(\alpha)} - P, \Phi \circ \pi_n \rangle| + 2 \sup_{Q \in \mathcal{P}_{\theta,r}^{(K,R)}} \langle Q, |\Phi - \Phi \circ \pi_n| \rangle. &(3.1.85)
\end{aligned}$$

Since the test function $\Phi \circ \pi_n$ is local and bounded on Γ_r and $P^{(\alpha)} \to P$, we have $|\langle P^{(\alpha)} - P, \Phi \circ \pi_n \rangle| \to 0$.

We now show that $\Phi \circ \pi_n \to \Phi$ uniformly on Γ_r, so proving the result. For $\omega \in \Gamma_r$,

$$\begin{aligned}
\Phi(\omega) - \Phi \circ \pi_n(\omega) &= \sum_{x \in \xi \cap U} \sum_{y \in \xi \cap \Lambda_n^c} T_{x,y}(\omega) &(3.1.86)\\
&\leq \frac{1}{2} \beta K^2 \sum_{x \in \xi \cap U} \sum_{y \in \xi \cap \Lambda_n^c} \sup_{s \geq |x-y| - 2R} v(s). &(3.1.87)
\end{aligned}$$

Since $|x-y| \geq {}^1\!/r$ for any distinct $x, y \in \xi$, the upper bound is not larger than

$$\beta K^2 C_{r,R,d} \int_n^\infty t^{d-1} \sup_{s \geq t-2R} v(s) \,\mathrm{d}t, \tag{3.1.88}$$

for some constant $C_{r,R,d}$ that depends only on r, R and d. Since this integral is finite, this bound vanishes in the n-limit.

□

In the following we use the assumption that $\liminf_{r\to 0} v(r) > 0$.

Lemma 3.1.19. *Fix K, R and let $P \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$ with $\langle P, \Phi\rangle < +\infty$. Then*

$$\left\langle P, N_U^2 \right\rangle < +\infty \qquad \text{and} \qquad \left\langle P, \left(N_U^{(\ell)}\right)^2 \right\rangle < +\infty. \tag{3.1.89}$$

PROOF. Pick $R^* < {}^1\!/2$ such that $v(r) \geq \zeta$ for all $r \leq R^*$. Then because $v \geq 0$,

$$\langle P, \Phi\rangle \geq \frac{1}{2} \int P(\mathrm{d}\omega) \sum_{x,y \in \xi \cap \Lambda_{R^*/4}} \mathbb{1}\{x \neq y\} \int_0^\beta v\left(|f_x(s) - f_y(s)|\right) \mathrm{d}s. \tag{3.1.90}$$

Define, for any $\omega \in \Omega^{(K,R)}$ and $x \in \xi$,

$$\tau_x = \min\left\{\delta, \inf\left\{s \in [0,\beta] : |f_x(s) - x| > \frac{R^*}{4}\right\}\right\}. \tag{3.1.91}$$

Since $x, y \in \Lambda_{R^*/4}$,

$$\begin{aligned} \int_0^\beta v\left(|f_x(s) - f_y(s)|\right) \mathrm{d}s &\geq \int_0^{\min\{\tau_x,\tau_y\}} v\left(|f_x(s) - f_y(s)|\right) \mathrm{d}s & (3.1.92)\\ &\geq \zeta \min\{\tau_x, \tau_y\}. & (3.1.93)\end{aligned}$$

By inserting an indicator on the event $\{\tau_x = \tau_y = \delta\}$, we have the lower bound

$$\langle P, \Phi\rangle \geq \frac{\zeta\delta}{2} \int P(\mathrm{d}\omega) \# \left\{(x,y) \in \left(\xi \cap \Lambda_{R^*/4}\right)^2 : x \neq y, \tau_x = \tau_y = \delta\right\}. \tag{3.1.94}$$

Since the event $\{\tau_x = \delta\}$ is decreasing in δ, and $P(\tau_x = \delta) \to 1$ as $\delta \to 0$, the integrand term $\#\left\{(x,y) \in \left(\xi \cap \Lambda_{R^*/4}\right)^2 : x \neq y, \tau_x = \tau_y = \delta\right\}$ tends to the number of distinct pairs in $\xi \cap \Lambda_{R^*/4}$. Hence for sufficiently small $\delta > 0$ we have

$$\langle P, \Phi \rangle \geq \frac{\zeta\delta}{2} \int P\left(\mathrm{d}\omega\right) \# \left\{ (x, y) \in \left(\xi \cap \Lambda_{R^*/4}\right)^2 : x \neq y \right\} \geq \frac{\zeta\delta}{8} \left\langle P, N^2_{\Lambda_{R^*/4}} \right\rangle . \tag{3.1.95}$$

Hence if $\langle P, \Phi \rangle < \infty$, the shift invariance of P implies that $\langle P, N_\Lambda^2 \rangle$ is finite for any bounded box Λ.

Since P is concentrated on configurations with leg length less than or equal to K, the expectation $\left\langle P, \left(N_\Lambda^{(\ell)}\right)^2 \right\rangle \leq K^2 \langle P, N_\Lambda^2 \rangle < \infty$.

□

In the following proof it will be useful to denote

$$\psi_R\left(t\right) = \begin{cases} \sup_{s \geq t-2R} v\left(s\right) & : t \geq 3R, \\ v\left(R\right) & : t \in [0, 3R)\,. \end{cases} \tag{3.1.96}$$

The assumptions on v imply that $\psi_R\left(t\right) = O\left(r^{-h}\right)$ for some $h > d$.

Lemma 3.1.20. *Fix $K, R \in \mathbb{N}$ and $\epsilon > 0$. Then for any $P \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$ such that $I^{(K,R)}\left(P\right) + \beta W_\mu\left(P\right) + \langle P, \Phi \rangle < \infty$ and any neighbourhood V of P in $\mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$, there exists an ergodic measure $\tilde{P} \in V$ and some $r > 0$ such that*

$$\tilde{P}\left(\Gamma_r\right) = 1, \tag{3.1.97}$$

$$W_\mu\left(\tilde{P}\right) \leq W_\mu\left(P\right) + \epsilon, \tag{3.1.98}$$

$$\left\langle \tilde{P}, \Phi \right\rangle \leq \langle P, \Phi \rangle + \epsilon, \tag{3.1.99}$$

$$I^{(K,R)}\left(\tilde{P}\right) \leq I^{(K,R)}\left(P\right) + \epsilon. \tag{3.1.100}$$

Proof. This is similar to [Geo94, Lemma 5.1]. Recall that P_n denotes the projection of P on Ω_n, the configuration space with anchors in the box $\Lambda_{2n} = [-n, n]^d$. Since $\langle P, \Phi \rangle < \infty$ (recall that W_μ is bounded below) and $\Phi \geq 0$, we have $\langle P_n, \Phi \rangle < \infty$. Hence $\lim_{r\to\infty} P_n\left(\Gamma_r\right) = 1$ for any fixed $n \in \mathbb{N}$. Therefore we can choose a sequence $r\left(n\right) \to \infty$ such that $\lim_{n\to\infty} P_n\left(\Gamma_{r(n)}\right) = 1$. Set $m = n + 3R$, and denote by $\hat{P}^{(n)}$ the probability measure under which the particle configurations in the (up to the boundary, disjoint) boxes $\Lambda_m + 2mk$, with $k \in \mathbb{Z}^d$, are independent and distributed as $P'_n := P_n\left(\cdot|\Gamma_{r(n)}\right)$. In particular, no points are contained in the corridors $(\Lambda_m \setminus \Lambda_n) + 2mk$.

Now let

$$P^{(n)} = \frac{1}{|\Lambda_m|}\int_{\Lambda_m} \hat{P}^{(n)} \circ \theta_z \mathrm{d}z. \tag{3.1.101}$$

It is clear that $P^{(n)} \in \mathcal{P}_\theta$, and a standard argument shows that $P^{(n)}$ is ergodic (see, for example, [Geo11, Theorem 14.12]). Since $\Gamma_{r(n)}$ is shift-invariant and $\hat{P}^{(n)}\left(\Gamma_{r(n)}\right) = 1$, it follows that $P^{(n)}\left(\Gamma_{r(n)}\right) = 1$ as well. We now show that for n sufficiently large, $\tilde{P} = P^{(n)}$ satisfies our requirements. We have to show

(1):

$$\limsup_{n\to\infty} I^{(K,R)}\left(P^{(n)}\right) \leq I^{(K,R)}\left(P\right), \tag{3.1.102}$$

(2):

$$\limsup_{n\to\infty} \left\langle P^{(n)}, N_U^{(\ell)}\right\rangle \leq \left\langle P, N_U^{(\ell)}\right\rangle, \tag{3.1.103}$$

(3):

$$\limsup_{n\to\infty} \left\langle P^{(n)}, \Phi\right\rangle \leq \left\langle P, \Phi\right\rangle, \tag{3.1.104}$$

(4):

$$P^{(n)} \to P \text{ in the topology } \tau_{\mathcal{L}}. \tag{3.1.105}$$

The proof of *(1)* can be found in the proof of [Geo94, Lemma 5.1].

For *(2)*, we first calculate

$$\left\langle P^{(n)}, N_U^{(\ell)}\right\rangle = \frac{1}{|\Lambda_m|}\int_{\Lambda_m} \mathrm{d}z \int \hat{P}^{(n)}\left(\mathrm{d}\omega\right) \sum_{x\in\xi\cap(U-z)} \ell\left(f_x\right). \tag{3.1.106}$$

Since we have the empty corridors around Λ_n, each of these x belong to Λ_n, and the measure $\hat{P}^{(n)}$ can be replaced by P'_n. Now we estimate the integration with respect to P'_n to integration with respect to $P\left(\cdot\right)/P_n\left(\Gamma_{r(n)}\right)$, and get

$$\left\langle P^{(n)}, N_U^{(\ell)}\right\rangle \leq \frac{1}{P_n\left(\Gamma_{r(n)}\right)|\Lambda_m|}\int_{\Lambda_m} \mathrm{d}z \int P\left(\mathrm{d}\omega\right) \sum_{x\in\xi\cap(U-z)} \ell\left(f_x\right). \tag{3.1.107}$$

The shift invariance of P and $\lim_{n\to\infty} P_n\left(\Gamma_{r(n)}\right) = 1$ implies that this bound approaches $\langle P, N_U^{(\ell)} \rangle$.

The proof of *(3)* begins similarly. First note that

$$\langle P^{(n)}, \Phi \rangle = \frac{1}{|\Lambda_m|} \int_{\Lambda_m} \mathrm{d}z \int \hat{P}^{(n)}(\mathrm{d}\omega) \sum_{x \in \xi \cap (U-z)} \sum_{y \in \xi} T_{x,y}(\omega). \tag{3.1.108}$$

The sum over y will be split into the sum over $y \in \xi \cap \Lambda_n$ and the remainder. For the first part, note that since x, y both belong to Λ_n, the measure $\hat{P}^{(n)}$ can be replaced by P_n'. Furthermore, since $T_{x,y}(\omega) \geq 0$, the integration with respect to P_n' may be estimated against the integration with respect to $P(\cdot)/P_n\left(\Gamma_{r(n)}\right)$. This produces

$$\begin{aligned} &\frac{1}{|\Lambda_m|} \int_{\Lambda_m} \mathrm{d}z \int \hat{P}^{(n)}(\mathrm{d}\omega) \sum_{x \in \xi \cap (U-z)} \sum_{y \in \xi \cap \Lambda_n} T_{x,y}(\omega) \\ &\qquad \leq \frac{1}{P_n\left(\Gamma_{r(n)}\right)} \frac{1}{|\Lambda_m|} \int_{\Lambda_m} \mathrm{d}z \int P(\mathrm{d}\omega) \sum_{x \in \xi \cap (U-z)} \sum_{y \in \xi} T_{x,y}(\omega). \end{aligned} \tag{3.1.109}$$

Once again. the shift invariance of P and the limit $P_n\left(\Gamma_{r(n)}\right) = 1$ implies that this bound approaches $\langle P, \Phi \rangle$.

We now address the sum over $y \in \xi \cap \Lambda_m^c$. Now $|x - y| \geq 3R$, so

$$T_{x,y}(\omega) \leq \beta K^2 \psi_R(|x-y|) \leq \beta K^2 \sup_{x:|x| \leq |z|+1} \psi_R(|x-y|) \leq \beta K^2 \psi_R(|y| - |z| - 1), \tag{3.1.110}$$

where in the last inequality we used $|x - y| \geq |x| - |y|$ and that ψ_R is non-increasing. Now we identify in which of the boxes $\Lambda_n + 2km$, with $k \in \mathbb{Z}^d$, the point y resides. Recall that each of these boxes is independently distributed. Hence for any $z \in \Lambda_m$,

$$\int \hat{P}^{(n)}(\mathrm{d}\omega) \sum_{x \in \xi \cap (U-z)} \sum_{y \in \xi \cap \Lambda_m^c} T_{x,y}(\omega) \tag{3.1.111}$$

$$\leq \beta K^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \int_{\Omega_n} P_n'(\mathrm{d}\omega^{(1)}) \int_{\Omega_n} P_n'(\mathrm{d}\omega^{(2)}) \,\#\left\{\xi^{(1)} \cap (U-z)\right\}$$

$$\times \sum_{y \in \left(\xi^{(2)} \cap \Lambda_n\right) + 2km} \psi_R(|y| - |z| - 1) \tag{3.1.112}$$

$$\leq \frac{\beta K^2}{P_n\left(\Gamma_{r(n)}\right)^2} \langle P, N_U \rangle \langle P, N_{\Lambda_n} \rangle \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \psi_R(2|k|m - m - |z| - 1), \tag{3.1.113}$$

where we estimated integrals with respect to P_n' against integrals with respect to the finite measure $P/P_n\left(\Gamma_{r(n)}\right)$ twice and used the shift invariance of P. Now from our tail condition on v, there exists a constant C (depending only upon R) such that $\psi_R\left(t\right) \leq Ct^{-h}$ for all $t \geq 0$. Hence

$$\begin{aligned}
&\int \hat{P}^{(n)}\left(\mathrm{d}\omega\right) \sum_{x\in\xi\cap(U-z)} \sum_{y\in\xi\cap\Lambda_m^c} T_{x,y}\left(\omega\right) \\
&\qquad \leq \frac{\beta K^2 C 2^d}{P_n\left(\Gamma_{r(n)}\right)^2} \left\langle P, N_U\right\rangle^2 n^d \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \left(2|k|m - m - |z| - 1\right)^{-h}. \qquad (3.1.114)
\end{aligned}$$

If we now pick some sequence $l = l\left(n\right)$ such that $l \sim n$ but $n^d\left(n-l\right)^{-h} \to 0$ as $n \to \infty$, and split the integral on $z \in \Lambda_m$ into the integrals on $z \in \Lambda_l$ and on the remainder. Then it follows that

$$\lim_{n\to\infty} \frac{1}{|\Lambda_m|} \int_{\Lambda_m} \mathrm{d}z \int \hat{P}^{(n)}\left(\mathrm{d}\omega\right) \sum_{x\in\xi\cap(U-z)} \sum_{y\in\xi\cap\Lambda_m^c} T_{x,y}\left(\omega\right) = 0. \qquad (3.1.115)$$

This proves *(3)*.

To prove *(4)*, we begin by picking $f \in \mathcal{L}$. By using an affine transformation, we may assume that $f = f\left(\cdot \cap \Delta\right)$ and $|f| \leq N_\Delta$ for some bounded measurable $\Delta \subset \mathbb{R}^d$. Now

$$\begin{aligned}
&\left|\left\langle P^{(n)}, f\right\rangle - \left\langle P, f\right\rangle\right| \\
&\qquad \leq \frac{1}{|\Lambda_m|} \int_{\Lambda_m} \mathrm{d}x \mathbb{1}\left\{x + \Delta \subset \Lambda_m\right\} \left| \frac{\left\langle P_n, f\circ\theta_x \mathbb{1}\left\{\Gamma_{r(n)}\right\}\right\rangle}{P_n\left(\Gamma_{r(n)}\right)} - \left\langle P, f\circ\theta_x\right\rangle \right| \\
&\qquad\quad + \frac{1}{|\Lambda_m|} \int_{\Lambda_m} \mathrm{d}x \mathbb{1}\left\{x + \Delta \not\subset \Lambda_m\right\} \left| \left\langle \hat{P}^{(n)}, N_{\Delta+x}\right\rangle + \left\langle P, N_{\Delta+x}\right\rangle \right|. \qquad (3.1.116)
\end{aligned}$$

Now $\left\langle P, N_{\Delta+x}\right\rangle = |\Delta|\mu\left(P\right) \leq |\Delta|\mu\left(P\right)/P_n\left(\Gamma_{r(n)}\right)$, where $\mu\left(P\right) < \infty$ is the intensity of P. Also,

$$\left\langle \hat{P}^{(n)}, N_{\Delta+x}\right\rangle = \frac{\left\langle P_n, N_{\Delta+x \pmod{2m+1}} \mathbb{1}\left\{\Gamma_{r(n)}\right\}\right\rangle}{P_n\left(\Gamma_{r(n)}\right)} \leq \frac{|\Delta|\mu\left(P\right)}{P_n\left(\Gamma_{r(n)}\right)}. \qquad (3.1.117)$$

Hence the second term on the right-hand side of (3.1.116) is not larger than the volume of $\{x \in \Lambda_m : x + \Delta \not\subset \Lambda_m\}$ (that is, surface order of Λ_m) times $O\left({}^1/_{|\Lambda_m|}\right)$. It vanishes in the limit. For the first term in (3.1.116), we bound

$$\begin{aligned} \left|P_n\left(f \circ \theta_x | \Gamma_{r(n)}\right) - P\left(f \circ \theta_x\right)\right| \leq & \left|\frac{1}{P_n\left(\Gamma_{r(n)}\right)} - 1\right| P_n\left(N_{\Delta+x}; \Gamma_{r(n)}\right) \\ & + P_n\left(N_{\Delta+x}; \Gamma^c_{r(n)}\right) \qquad (3.1.118) \\ \leq & |\Delta| \mu\left(P\right) \left|\frac{1}{P_n\left(\Gamma_{r(n)}\right)} - 1\right| \\ & + P\left(N^2_\Delta\right)^{\frac{1}{2}} \left(1 - P_n\left(\Gamma_{r(n)}\right)\right)^{\frac{1}{2}}. \qquad (3.1.119) \end{aligned}$$

By Lemma 3.1.19, $P\left(N^2_\Delta\right) < \infty$, and the required terms vanish in the $n \to \infty$ limit. Hence *(4)* holds.

$\square$

Fix $K, R \in \mathbb{N}$, $\epsilon > 0$ and pick $P \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$ such that $I_\alpha\left(P\right) + \beta W_\mu\left(P\right) + \langle P, \Phi\rangle < \infty$. By Lemma 3.1.20, we may fix $r > 0$ and ergodic measure $\tilde{P} \in \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$ such that $\tilde{P}\left(\Gamma_r\right) = 1$, $\left\langle \tilde{P}, \Phi\right\rangle \leq \langle P, \Phi\rangle + \epsilon$, $\beta W_\mu\left(\tilde{P}\right) \leq \beta W_\mu\left(P\right) + \epsilon$, and $I^{(K,R)}_\alpha\left(\tilde{P}\right) \leq I^{(K,R)}_\alpha\left(P\right) + \epsilon$.

Because Φ and W_μ are bounded below, $I_\alpha\left(P\right) + \beta W_\mu\left(P\right) + \langle P, \Phi\rangle < \infty$ implies that $I^{(K,R)}_\alpha\left(\tilde{P}\right) < \infty$, and so for L sufficiently large there is a density $f^{(K,R)}_L$ of $\tilde{P}_L$ with respect to $\mathtt{Q}^{(K,R)}_{\alpha,L}$. We think of $\mathfrak{R}^{(K,R)}_L$ as a map $\Omega_L \to \mathcal{P}_\theta\left(\Omega^{(K,R)}\right)$. Now define the event

$$\begin{aligned} C_L = \Big\{\omega \in \Omega^{(K,R)} : \beta W_\mu\left(\mathfrak{R}^{(K,R)}_L\right) \leq \beta W_\mu\left(\tilde{P}\right) + \epsilon, \left\langle \mathfrak{R}^{(K,R)}_L, \Phi\right\rangle \leq \left\langle \tilde{P}, \Phi\right\rangle + \epsilon, \\ \frac{1}{|\Lambda_L|} \log f^{(K,R)}_L\left(\omega\right) \leq I^{(K,R)}_\alpha\left(\tilde{P}\right) + \epsilon\Big\}. \qquad (3.1.120) \end{aligned}$$

Now we can estimate

$$
\begin{aligned}
\mathtt{E}_{\alpha}^{(K,R)} & \left[\mathrm{e}^{-|\Lambda_L|\left(\beta W_\mu\left(\mathfrak{R}_L^{(K,R)}\right)+\left\langle \mathfrak{R}_L^{(K,R)},\Phi\right\rangle\right)} \right] \\
&= \int_{\Omega_L^{(K,R)}} \mathrm{d}\mathtt{Q}_{\alpha,L}^{(K,R)} \mathrm{e}^{-|\Lambda_L|\left(\beta W_\mu\left(\mathfrak{R}_L^{(K,R)}\right)+\left\langle \mathfrak{R}_L^{(K,R)},\Phi\right\rangle\right)} && (3.1.121)\\
&\geq \int_{C_L} \tilde{P}_L\left(\mathrm{d}\omega\right) \frac{1}{f_L^{(K,R)}\left(\omega\right)} \mathrm{e}^{-|\Lambda_L|\left(\beta W_\mu\left(\mathfrak{R}_L^{(K,R)}\right)+\left\langle \mathfrak{R}_L^{(K,R)},\Phi\right\rangle\right)} && (3.1.122)\\
&\geq \mathrm{e}^{-|\Lambda_L|\left(I_\alpha^{(K,R)}\left(\tilde{P}\right)+\beta W_\mu\left(\tilde{P}\right)+\left\langle \tilde{P},\Phi\right\rangle+3\epsilon\right)} \tilde{P}_L\left(C_L\right). && (3.1.123)
\end{aligned}
$$

Now the continuity of the map $P \mapsto W_\mu\left(P\right) + \langle P, \Phi\rangle$, the law of large numbers and McMillan's Ergodic Theorem imply that

$$
\lim_{L\to\infty} \tilde{P}_L\left(C_L\right) = 1. \tag{3.1.124}
$$

The form of McMillan's Ergodic Theorem for our purposes can be found in [Fri70; NZ79], and is used in a similar way to us by [GZ93]. It follows from (3.1.123) and (3.1.124) that

$$
\begin{aligned}
\liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{E}_{\alpha}^{(K,R)} & \left[\mathrm{e}^{-|\Lambda_L|\left(\beta W_\mu\left(\mathfrak{R}_L^{(K,R)}\right)+\left\langle \mathfrak{R}_L^{(K,R)},\Phi\right\rangle\right)} \right] \\
&\geq -I_\alpha^{(K,R)}\left(\tilde{P}\right) - \beta W_\mu\left(\tilde{P}\right) - \left\langle \tilde{P}, \Phi\right\rangle - 3\epsilon && (3.1.125)\\
&\geq -I_\alpha^{(K,R)}\left(P\right) - \beta W_\mu\left(P\right) - \left\langle P, \Phi\right\rangle - 6\epsilon. && (3.1.126)
\end{aligned}
$$

Now let $\epsilon \to 0$ and take the infimum over P. With Lemma 3.1.16, we get

$$
\begin{aligned}
\liminf_{L\to\infty} & \frac{1}{|\Lambda_L|} \log \mathtt{E}_\alpha \left[\mathrm{e}^{\beta\mu N_{\Lambda_L}^{(\ell)} - \beta V_{\Lambda_L,\mu} - H_{\Lambda_L}} \right] \\
&\geq - \limsup_{K\to\infty} \limsup_{R\to\infty} \inf_{P\in\mathcal{P}_\theta\left(\Omega^{(K,R)}\right)} \left\{ I_\alpha^{(K,R)}\left(P\right) + \beta W_\mu\left(P\right) + \left\langle P, \Phi\right\rangle \right\}. && (3.1.127)
\end{aligned}
$$

Lemma 3.1.21.

$$
\begin{aligned}
\limsup_{K\to\infty} \limsup_{R\to\infty} & \inf_{P\in\mathcal{P}_\theta\left(\Omega^{(K,R)}\right)} \left\{ I_\alpha^{(K,R)}\left(P\right) + \beta W_\mu\left(P\right) + \left\langle P, \Phi\right\rangle \right\} \\
&\leq \inf_{P\in\mathcal{P}_\theta} \left\{ I_\alpha\left(P\right) + \beta W_\mu\left(P\right) + \left\langle P, \Phi\right\rangle \right\} && (3.1.128)
\end{aligned}
$$

Proof. Fix $P \in \mathcal{P}_\theta$ such that $I_\alpha(P) + \beta W_\mu(P) + \langle P, \Phi \rangle < \infty$. For $K, R \in \mathbb{N}$ consider $P_{K,R} = P \circ \pi_{K,R}^{-1}$. Then $\Phi \geq \Phi \circ \pi_{K,R}$ implies that $\langle P_{K,R}, \Phi \rangle \leq \langle P.\Phi \rangle$. Since $\mathcal{G}_\mu$ is non-decreasing and $N_U^{(\ell)} \geq N_U^{(\ell)} \circ \pi_{K,R}$, we also have $W_\mu(P_{K,R}) \leq W_\mu(P)$. Finally, by (3.1.64), we have $I^{(K,R)}(P_{K,R}) \leq I(P)$.

$\square$

This concludes the proof of Theorem 3.1.1.

3.2 Equivalence of Ensembles

From Corollary 3.1.3 we have an expression for the thermodynamic pressure, and in [ACK11] we find an expression for the thermodynamic free energy for some parameter values. Therefore we are in a position to discuss equivalence of ensembles at the level of thermodynamic functions. If we have this equivalence, then we expect the free energy to be given by the Legendre transform of the pressure.

For the derivation of the free energy in [ACK11], we need the additional condition on the pair potential v that the integral

$$E(v) = \int_{\mathbb{R}^d} v(|x|)\, \mathrm{d}x \tag{3.2.1}$$

is finite. In particular, this excludes some potentials that we were able to consider in our previous discussion, such as the hard-sphere potential. We also denote with $\varrho^* = \varrho^*(\beta)$ the unique solution of the equation

$$\frac{1}{(4\pi\beta)^{\frac{d}{2}}} = \varrho^* \mathrm{e}^{\beta\varrho^* E(v)}. \tag{3.2.2}$$

Note that, since $v \geq 0$ everywhere, we have $\varrho^* \leq \varrho_c$.

Theorem 3.2.1. *Fix $\beta > 0$ and $x \leq \varrho^*(\beta)$. Then the non-stabilised free energy is given by*

$$f^{(v)}(\beta, x) = -p(\beta, 0) + \frac{1}{\beta} \inf_{P \in \mathcal{P}_\theta : \left\langle P, N_U^{(\ell)} \right\rangle \leq x} \left\{ I_0(P) + \langle P, \Phi \rangle \right\}. \tag{3.2.3}$$

PROOF. This is [ACK11, Corollary 1.3].

□

Recall that Corollary 3.1.3 gives the thermodynamic pressure of the non-stabilised model with $\beta > 0$ and $\alpha \leq 0$ as

$$p^{(v)}(\beta, \alpha) = p(\beta, \alpha) - \frac{1}{\beta} \inf_{P \in \mathcal{P}_\theta} \{I_\alpha(P) + \langle P, \Phi \rangle\}. \tag{3.2.4}$$

To relate $f^{(v)}$ and $p^{(v)}$, it will prove useful to relate the rate functions I_α and I_0.

Lemma 3.2.2. *Let $\alpha \leq 0$. Then for any $P \in \mathcal{P}_\theta$,*

$$I_\alpha(P) = I_0(P) - \beta\alpha \left\langle P, N_U^{(\ell)} \right\rangle + \beta p(\beta, \alpha) - \beta p(\beta, 0). \tag{3.2.5}$$

PROOF. We use the expression for I_α given in (2.2.14). First note from [Tak90], that $\mathsf{Q}_{0,\Lambda}$ and $\mathsf{Q}_{\alpha,\Lambda}$ are mutually absolutely continuous. This follows because the associated intensity measures are mutually absolutely continuous and have finite Hellinger distance. Let $\rho_{\alpha,\Lambda}$ denote the intensity measure of $\mathsf{Q}_{\alpha,\Lambda}$, and $\rho_{0,\Lambda}$ the intensity measure of $\mathsf{Q}_{0,\Lambda}$. Then

$$\phi(x, f) := \frac{\mathrm{d}\rho_{0,\Lambda}}{\mathrm{d}\rho_{\alpha,\Lambda}}(x, f) = \exp(-\beta\alpha\ell(f)). \tag{3.2.6}$$

The Hellinger distance is then given by

$$\begin{aligned} d(\rho_{0,\Lambda}, \rho_{\alpha,\Lambda})^2 &= \frac{1}{2} \int_{\Lambda \times E} |\sqrt{\phi} - 1|^2 \mathrm{d}\rho_{\alpha,\Lambda} \\ &= |\Lambda| \sum_{k \in \mathbb{N}} q_k e^{\beta\alpha k} \left(\exp\left(-\frac{1}{2}\beta\alpha k \right) - 1 \right)^2 < |\Lambda| \beta p(\beta, 0) < +\infty. \end{aligned} \tag{3.2.7}$$

This mutual absolute continuity implies that P_Λ is absolutely continuous with respect to $\mathsf{Q}_{0,\Lambda}$ if and only if P_Λ is absolutely continuous with respect to $\mathsf{Q}_{\alpha,\Lambda}$.

Now given the empirical cycle density $\boldsymbol{\lambda}^{(N)}$, the measures $\mathsf{Q}_{0,\Lambda}$ and $\mathsf{Q}_{\alpha,\Lambda}$ are equal and the random variables $\{\mathcal{N}_{\Lambda,k}\}_{k \in \mathbb{N}}$ are all independent in both measures.

Therefore given $\omega' \in \Omega$,

$$\frac{\mathrm{d}\mathsf{Q}_{0,\Lambda}}{\mathrm{d}\mathsf{Q}_{\alpha,\Lambda}}\left(\omega'\right) = \prod_{k\in\mathbb{N}} \frac{\mathsf{Q}_{0,\Lambda}\left(\mathcal{N}_{\Lambda,k} = \left|\Lambda\right|\boldsymbol{\lambda}_k^{(N)}\left(\omega'\right)\right)}{\mathsf{Q}_{\alpha,\Lambda}\left(\mathcal{N}_{\Lambda,k} = \left|\Lambda\right|\boldsymbol{\lambda}_k^{(N)}\left(\omega'\right)\right)} \tag{3.2.8}$$

$$= \exp\left(-\beta\alpha N_\Lambda^{(\ell)}\left(\omega'\right) + \left|\Lambda\right|\beta\left(p\left(\beta,\alpha\right) - p\left(\beta,0\right)\right)\right). \tag{3.2.9}$$

If P is absolutely continuous with respect to the two reference measures,

$$I_\alpha\left(P\right) - I_0\left(P\right) = \lim_{\left|\Lambda\right|\to\infty} \frac{1}{\left|\Lambda\right|}\left(\int \frac{\mathrm{d}P_\Lambda}{\mathrm{d}\mathsf{Q}_{\alpha,\Lambda}}\log\frac{\mathrm{d}P_\Lambda}{\mathrm{d}\mathsf{Q}_{\alpha,\Lambda}}\mathrm{d}\mathsf{Q}_{\alpha,\Lambda} - \int \frac{\mathrm{d}P_\Lambda}{\mathrm{d}\mathsf{Q}_{0,\Lambda}}\log\frac{\mathrm{d}P_\Lambda}{\mathrm{d}\mathsf{Q}_{0,\Lambda}}\mathrm{d}\mathsf{Q}_{0,\Lambda}\right) \tag{3.2.10}$$

$$= \lim_{\left|\Lambda\right|\to\infty} \frac{1}{\left|\Lambda\right|}\int \mathrm{d}P_\Lambda \log\frac{\mathrm{d}P_\Lambda/\mathrm{d}\mathsf{Q}_{\alpha,\Lambda}}{\mathrm{d}P_\Lambda/\mathrm{d}\mathsf{Q}_{0,\Lambda}} \tag{3.2.11}$$

$$= \lim_{\left|\Lambda\right|\to\infty} \frac{1}{\left|\Lambda\right|}\int \mathrm{d}P_\Lambda \log\frac{\mathrm{d}\mathsf{Q}_{0,\Lambda}}{\mathrm{d}\mathsf{Q}_{\alpha,\Lambda}} \tag{3.2.12}$$

$$= -\beta\alpha \lim_{\left|\Lambda\right|\to\infty} \frac{1}{\left|\Lambda\right|}\left\langle P_\Lambda, N_\Lambda^{(\ell)}\right\rangle + \beta p\left(\beta,\alpha\right) - \beta p\left(\beta,0\right) \tag{3.2.13}$$

$$= -\beta\alpha\left\langle P, N_U^{(\ell)}\right\rangle + \beta p\left(\beta,\alpha\right) - \beta p\left(\beta,0\right). \tag{3.2.14}$$

Otherwise, both rate functions equal $+\infty$.

□

Theorem 3.2.3. *For $x \leq \varrho^*$,*

$$f^{(v)}\left(\beta,x\right) = \sup_{\alpha\leq 0}\left\{\alpha x - p^{(v)}\left(\beta,\alpha\right)\right\}. \tag{3.2.15}$$

Proof. From Lemma 3.2.2, we find

$$\sup_{\alpha\leq 0}\left\{\alpha x - p^{(v)}\left(\beta,\alpha\right)\right\} = \sup_{\alpha\leq 0}\left\{\alpha x - p\left(\beta,\alpha\right) + \frac{1}{\beta}\inf_P\left\{I_\alpha\left(P\right) + \left\langle P,\Phi\right\rangle\right\}\right\} \tag{3.2.16}$$

$$= -p\left(\beta,0\right) + \frac{1}{\beta}\sup_{\alpha\leq 0}\inf_P\left\{I_0\left(P\right) + \left\langle P,\Phi\right\rangle + \beta\alpha\left(x - \left\langle P, N_U^{(\ell)}\right\rangle\right)\right\}. \tag{3.2.17}$$

For $\alpha \leq 0$ and $P \in \mathcal{P}_\theta$, denote

$$F_\alpha\left(P\right) = I_0\left(P\right) + \left\langle P,\Phi\right\rangle + \beta\alpha\left(x - \left\langle P, N_U^{(\ell)}\right\rangle\right). \tag{3.2.18}$$

Then we rewrite the variational expression as

$$\sup_{\alpha\leq 0}\inf_{P}\left\{F_\alpha(P)\right\}=\min\left\{\sup_{\alpha\leq 0}\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{F_\alpha(P)\right\},\sup_{\alpha\leq 0}\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle> x}\left\{F_\alpha(P)\right\}\right\}. \tag{3.2.19}$$

Note that on $\left\{P:\left\langle P,N_U^{(\ell)}\right\rangle>x\right\}$, the function F_α is pointwise decreasing in α, whereas on $\left\{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x\right\}$, it is pointwise non-decreasing. Therefore

$$\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle> x}\left\{F_\alpha(P)\right\}\text{ is non-increasing in }\alpha, \tag{3.2.20}$$

$$\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{F_\alpha(P)\right\}\text{ is non-decreasing in }\alpha. \tag{3.2.21}$$

This means that the optimal value of α for the $\left\{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x\right\}$ case is $\alpha=0$.

$$\sup_{\alpha\leq 0}\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{F_\alpha(P)\right\}=\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{F_0(P)\right\}. \tag{3.2.22}$$

For all $P'\in\mathcal{P}_\theta$ such that $\left\langle P',N_U^{(\ell)}\right\rangle>x$,

$$\alpha<\frac{F_0(P')-\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{F_0(P)\right\}}{\beta\left(\left\langle P',N_U^{(\ell)}\right\rangle-x\right)}\implies F_\alpha(P')>\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{F_0(P)\right\}. \tag{3.2.23}$$

Hence

$$\sup_{\alpha\leq 0}\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle> x}\left\{F_\alpha(P)\right\}\geq\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{F_0(P)\right\}, \tag{3.2.24}$$

and

$$\sup_{\alpha\leq 0}\left\{\alpha x-p^{(v)}(\beta,\alpha)\right\}=-p(\beta,0)+\frac{1}{\beta}\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{I_0(P)+\langle P,\Phi\rangle\right\}. \tag{3.2.25}$$

This is the expression for $f^{(v)}$ given in Theorem 3.2.1.

□

Remark 3.2.4. *The expression* $\sup_{\alpha\leq 0}\left\{\alpha x-p^{(v)}(\beta,\alpha)\right\}$ *exists and is finite for* $x>\varrho^*$*, but [ACK11] does not prove the equality with the free energy in this regime. In Theorem 3.2.5, we will use* $f^{(v)}$ *to denote this expression even for* $x>\varrho^*$*.* ◇

Now we consider a stabilised model. In particular, we set

$$U_\mu(x) = \frac{a}{2}x^2 - \frac{(\mu - ax)_+^2}{2a}. \tag{3.2.26}$$

This is inspired by the PM interaction energy, with the minimal change to ensure that $U_\mu(x) - \mu x$ is non-decreasing in x. In Lemma 4.3.10 we will find that this change is required to get the lower semicontinuous regularisation of the PM interaction energy in the empirical cycle count framework.

Theorem 3.2.5. *For $x \geq 0$ and U_μ given by* (3.2.26)*,*

$$\sup_{\alpha \leq 0, \mu \in \mathbb{R}} \left\{(\alpha + \mu)x - p_U^{(v)}(\beta, \alpha, \mu)\right\} = f^{(v)}(\beta, x) + \frac{a}{2}x^2. \tag{3.2.27}$$

Proof. Our proof begins similarly to that of Theorem 3.2.3. Using Lemma 3.2.2 and (3.2.26) we find

$$\sup_{\alpha \leq 0, \mu \in \mathbb{R}} \left\{(\alpha + \mu)x - p_U^{(v)}(\beta, \alpha, \mu)\right\} = -p(\beta, 0) + \frac{1}{\beta} \sup_{\alpha \leq 0, \mu \in \mathbb{R}} \inf_P \left\{F_{\alpha,\mu}(P)\right\}, \tag{3.2.28}$$

where

$$\begin{aligned} F_{\alpha,\mu}(P) = I_0(P) + \langle P, \Phi \rangle + \beta(\alpha + \mu)\left(x - \left\langle P, N_U^{(\ell)} \right\rangle\right) + \frac{\beta a}{2}\left\langle P, N_U^{(\ell)} \right\rangle^2 \\ - \frac{\beta}{2a}\left(\mu - a\left\langle P, N_U^{(\ell)} \right\rangle\right)_+^2. \end{aligned} \tag{3.2.29}$$

The optimisation over α proceeds similarly to that in Theorem 3.2.3, so here we focus on the μ optimisation.

Given $P \in \mathcal{P}_\theta$, we have

$$\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}\mu}F_{\alpha,\mu}(P) = \begin{cases} x - \left\langle P, N_U^{(\ell)} \right\rangle & : \left\langle P, N_U^{(\ell)} \right\rangle \geq \frac{\mu}{a} \\ x - \frac{\mu}{a} & : \left\langle P, N_U^{(\ell)} \right\rangle \leq \frac{\mu}{a}. \end{cases} \tag{3.2.30}$$

This implies that on $\left\{P : \left\langle P, N_U^{(\ell)} \right\rangle > x\right\}$, $F_{\alpha,\mu}(P)$ is non-increasing in μ for all μ. In contrast, on $\left\{P : \left\langle P, N_U^{(\ell)} \right\rangle \leq x\right\}$, $F_{\alpha,\mu}(P)$ is non-increasing in μ for $\mu \geq ax$ but is non-decreasing in μ for $\mu \leq ax$. Therefore we arrive at

$$\sup_{\alpha\leq 0,\mu\in\mathbb{R}} \inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x} \left\{F_{\alpha,\mu}\left(P\right)\right\} = \inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x} \left\{F_{0,ax}\left(P\right)\right\}, \tag{3.2.31}$$

and by taking α, μ sufficiently negative we find

$$\sup_{\alpha\leq 0,\mu\in\mathbb{R}} \inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle> x} \left\{F_{\alpha,\mu}\left(P\right)\right\} \geq \inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x} \left\{F_{0,ax}\left(P\right)\right\}. \tag{3.2.32}$$

Now

$$F_{0,ax}\left(P\right) = I_0\left(P\right) + \left\langle P,\Phi\right\rangle + \frac{\beta}{2}ax^2, \tag{3.2.33}$$

and therefore

$$\begin{aligned}\sup_{\alpha\leq 0,\mu\in\mathbb{R}} &\left\{\left(\alpha+\mu\right)x - p_U^{(v)}\left(\beta,\alpha,\mu\right)\right\}\\ &= -p\left(\beta,0\right) + \frac{1}{\beta}\inf_{P:\left\langle P,N_U^{(\ell)}\right\rangle\leq x}\left\{I_0\left(P\right)+\left\langle P,\Phi\right\rangle\right\} + \frac{1}{2}ax^2,\end{aligned} \tag{3.2.34}$$

as required.

□

3.3 Remarks on Negative Interactions

The above arguments were for interaction potentials that are non-negative. I present here an incomplete plan for how to incorporate interactions with negative components into the above analysis and in particular how it needs to be developed. I omit here energetic chemical potentials and stabilising terms, because the difficulties persist even in this simpler case.

The plan begins by decomposing the interaction terms $v\left(r\right) = v^{(+)}\left(r\right) + v^{(-)}\left(r\right)$, where $v^{(+)} \geq 0$ and $v^{(-)} \leq 0$. We are able to derive a LDP for the Hamiltonian $H^{(+)}$ corresponding to $v^{(+)}$ by using Theorem 3.1.1. The associated measure has the Gibbs distribution

$$\mathtt{Q}_{\alpha}^{(+)}(\mathrm{d}\omega) = \frac{\mathrm{e}^{-H_{\Lambda}^{(+)}(\omega)}}{Z^{(+)}}\mathtt{Q}_{\alpha}(\mathrm{d}\omega). \tag{3.3.1}$$

Now we can use this measure as a base upon which we can apply the Hamiltonian $H^{(-)}$ corresponding to $v^{(-)}$. Because the corresponding $\Phi^{(-)}(\omega) \leq 0$ everywhere, the large deviation lower bound for $H^{(-)}$ proceeds similarly to the large deviation upper bound for $H^{(+)}$.

Let us define

$$\Phi^{(-,R,M,S)}(\omega) = \sum_{x \in \xi \cap U} \sum_{y \in \xi} T_{x,y}^{(-,R,M)}(\omega) \mathbb{1}\left\{N_{\Lambda_R}(\omega) \leq S\right\} \tag{3.3.2}$$

$$T_{x,y}^{(-,R,M)}(\omega) = \begin{cases} \max\left\{T_{x,y}^{(-)}(\omega), -M\right\} & : |x-y|_{\infty} \leq R \\ 0 & : |x-y|_{\infty} > R, \end{cases} \tag{3.3.3}$$

for $R, M, S \in [0, +\infty]$ and $\omega \in \Omega$.

Lemma 3.3.1. *Let $\tilde{\Phi}^{(-)}$ be the $R \to \infty$ pointwise limit of $\Phi^{(-,R,\infty,\infty)}$. Then*

$$\liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{E}_{\alpha}\left[\mathrm{e}^{-H_{\Lambda_L}^{(+)} - H_{\Lambda_L}^{(-)}}\right] \geq -\inf_{P \in \mathcal{P}_{\theta}} \left\{I(P) + \left\langle P, \Phi^{(+)} + \tilde{\Phi}^{(-)}\right\rangle\right\}. \tag{3.3.4}$$

The proof of this Lemma is given in Appendix B.

Whilst not precisely the expected form - the pointwise limit does not necessarily permit infinite-range behaviour - this is close enough for the moment. Let us turn our attention to the large deviation upper bound. Now we cannot mirror the argument for the large deviation lower bound for $H^{(+)}$, because we cannot just restrict our probability space with the indicator function - this gives the wrong bound.

To apply Varadhan's Lemma to get our upper bound we need to somehow 'make' the expectation $\langle P, \Phi^{(-)}\rangle$ lower semicontinuous and obey some boundedness condition (for example, (2.1.6)). Note that these conditions bear a resemblance to the *stable* and *superstable* conditions described in [Rue69] when considering interacting point processes (that is, without marks).

Definition 3.3.2. Let U_Λ be the interaction energy resulting from particles in the region Λ interacting with potential v. Let $N^{(\ell)}_\Lambda$ denote the number of these particles. Then the interaction v is said to be *stable* if there exists $B \geq 0$ such that

$$U_\Lambda \geq -BN^{(\ell)}_\Lambda, \tag{3.3.5}$$

for all Λ.

If there exists $B, C > 0$ such that

$$U_\Lambda \geq -BN^{(\ell)}_\Lambda + C\frac{\left(N^{(\ell)}_\Lambda\right)^2}{|\Lambda|}, \tag{3.3.6}$$

then the interaction is said to be *superstable*.

To highlight the comparison, note that (3.3.6) implies that the energy is bounded below, and that if we replace the 'particle number' in (3.3.5) with our 'anchor number' then we essentially have the bound required in Lemma 3.1.4 to have lower semicontinuity.

Whilst [Rue69] gives necessary and sufficient conditions on their potential for it to be stable, we have the extra complication of our random marks. In particular, we have the possibility for anchors arbitrarily far from the region Λ to interact with anchors in that box by having loops that travel arbitrarily far from the anchor. This not only has consequences for the locality condition, but allows for arbitrarily negative energies even in the overall potential, $v(r)$, is positive for r sufficiently large.

One toy model that could yield results is that of a hard-sphere interaction with an artificial bound on how far the loops can travel from the anchor. In particular, the self-interaction energy of a loop then places a restriction on the 'time' length of the mark - $\ell(f_x)$. Suppose that we have a hard-core with radius r_- and the loop is restricted to a ball of radius R. Then we force $\ell(f) \leq \left({}^{R}\!/_{r_-}\right)^d$, and our particle number is now a continuous function of the anchor number. This overcomes many of the difficulties of the full model and yet, whilst the hard-core is of physical interest, the travel bound feels artificial and forced.

Chapter 4

Large Deviations for Cycle Count Models

In this chapter we derive large deviation principles for the empirical cycle count under various interaction energies. First we derive a principle for the non-interacting (ideal) model in Section 4.1. This is done by using Baldi's Theorem and the independent Poisson structure of the reference process. In Section 4.2, we present the toy example of the Cycle-Mean-Field interaction model that proceeds via Varadhan's Lemma without the complications of the Particle-Mean-Field model. To overcome difficulties resulting from the lack of continuity of the PM tilt, in Section 4.3 we make use of lower semicontinuous regularisations and restricted probability spaces. This is done for the generalised version because the quadratic nature of the physical PM model is not itself significant. In Section 4.4 we find a large deviation principle for the full cycle HYL model. This builds directly upon the PM model and also uses lower semicontinuous regularisations. The behaviour of the minimisers of the rate functions of these large deviation principles is discussed in Chapter 5.

Also note that in this chapter, the inverse temperature $\beta > 0$ has little significance. For the sake of clarity in our proofs, we will set $\beta = 1$ and suppress it from the notation. Once we have these results, it is usually a matter of dimensional analysis to reintroduce it.

4.1 Ideal Bose Gas Model

Proposition 4.1.1 (Ideal Bose gas, grand canonical ensemble). *For $d \in \mathbb{N}$, $\beta > 0$, $\alpha \leq 0$ and $\mathrm{bc} \in \{\varnothing, \mathrm{Dir}\}$ (or $\alpha < 0$ and $\mathrm{bc} = \mathrm{per}$), the sequence $(\nu_{N,\alpha}^{(\mathrm{bc})})_{N\in\mathbb{N}}$ satisfies a LDP on $\ell_1(\mathbb{R}_+)$ with rate $|\Lambda_N|$ and rate function*

$$I_\alpha(x) = \sum_{k=1}^{\infty} x_k \left(\log \frac{x_k}{q_k \mathrm{e}^{\beta\alpha k}} - 1\right) + \overline{q}^{(\alpha)}, \qquad \forall x \in \ell_1(\mathbb{R}_+). \tag{4.1.1}$$

Remark 4.1.2. *The condition on α arises from the $\overline{q}^{(\alpha)}$ term. Our reference marked Poisson point process is superposition of independent marked Poisson point processes on $\mathbb{R}^d \times \mathcal{C}_k$ with intensity measure given in (2.2.8). The superposition is itself a marked Poisson point process if and only if $\sum_{k\in\mathbb{N}} q_{\Lambda,k}^{(\mathrm{bc})} \mathrm{e}^{\beta\alpha k} < \infty$. In Appendix A we show that this sum and its limit are finite in the $\mathrm{bc} \in \{\varnothing, \mathrm{Dir}\}$ cases if and only if $\alpha \leq 0$, and in the $\mathrm{bc} = \mathrm{per}$ case if and only if $\alpha < 0$.* $\diamond$

Our derivation of this LDP will be based on applying Baldi's Theorem (recall Lemma 2.1.14). We shall now set about establishing that the hypotheses of Baldi's Theorem are satisfied.

Lemma 4.1.3. *For $\alpha \leq 0$ and $\mathrm{bc} \in \{\varnothing, \mathrm{Dir}\}$ (or $\alpha < 0$ and $\mathrm{bc} = \mathrm{per}$), $\{\nu_{N,\alpha}\}_{N\in\mathbb{N}}$ is an exponentially tight sequence of measures.*

Proof. Recall the definition of exponential tightness given in Definition 2.1.12. We first consider the set

$$K_\Gamma = \{x \in \ell_1(\mathbb{R}_+) : f(x) \leq \Gamma\}, \tag{4.1.2}$$

$$f : \ell_1(\mathbb{R}_+) \to [0, +\infty], \qquad x \mapsto \sum_{k=1}^{\infty} x_k \log(k+1). \tag{4.1.3}$$

To prove that this set is compact, we prove that K_Γ is closed and totally bounded.

First note that f is lower semicontinuous by Fatou's Lemma. Therefore K_Γ is closed. To show total boundedness, fix $\varepsilon > 0$ and choose $N > \exp({}^{2\Gamma}/_{\varepsilon})$. Then define the subset

$$K_\Gamma^N = \{x \in K_\Gamma : x_j = 0 \quad \forall j > N\}. \tag{4.1.4}$$

This set is isomorphic to $\left\{x \in \mathbb{R}_+^N : \sum_{k=1}^N x_k \log(k+1) \leq \Gamma\right\}$, which is a closed and bounded subset of $\mathbb{R}^N$ and is therefore compact. This means that it is also totally bounded, and we have a finite set of points $\{y^{(1)}, \ldots, y^{(I)}\} \subset K_\Gamma^N$ such that their $\varepsilon/2$-open balls cover K_Γ^N. Given $x \in K_\Gamma$, let $x^{(N)} := (x_1, \ldots, x_N, 0, \ldots) \in K_\Gamma^N$. We now choose an appropriate $y^{(i)}$ such that $x^{(N)}$ is contained in the $\varepsilon/2$-open ball of $y^{(i)}$. Then

$$\begin{aligned} |x - y^{(i)}|_1 &= \sum_{k=1}^{N} |x_k - y_k^{(i)}| + \sum_{k=N+1}^{\infty} |x_k| && (4.1.5) \\ &\leq \frac{\varepsilon}{2} + \frac{\Gamma}{\log(N+1)} < \varepsilon, && (4.1.6) \end{aligned}$$

as required.

To show the required decay for the event K_Γ^c, we employ the Chernoff bound:

$$\nu_{N,\alpha}(K_\Gamma^c) = \nu_{N,\alpha}\left(\mathrm{e}^{tf\left(\boldsymbol{\lambda}^{(\Lambda_N)}\right)} > \mathrm{e}^{t\Gamma}\right) \leq \mathrm{e}^{-t\Gamma}\mathbb{E}_{\nu_{N,\alpha}}\left[\mathrm{e}^{tf\left(\boldsymbol{\lambda}^{(\Lambda_N)}\right)}\right] \qquad \forall t > 0. \quad (4.1.7)$$

If we choose $t = \varepsilon|\Lambda_N|$ for some $\varepsilon > 0$, then we find

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \nu_{N,\alpha}(K_\Gamma^c) &\leq -\varepsilon\Gamma + \limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \mathcal{L}_N^f(\varepsilon|\Lambda_N|) && (4.1.8) \\ &= -\varepsilon\Gamma + \mathcal{L}^f(\varepsilon), && (4.1.9) \end{aligned}$$

where $\mathcal{L}^f$ and $\mathcal{L}_N^f$ are the (limiting) logarithmic moment generating functions of the random variable $f\left(\boldsymbol{\lambda}^{(\Lambda_N)}\right)$. From this bound we can clearly prove exponential tightness if $\mathcal{L}^f(\varepsilon) < +\infty$.

Since $|\Lambda_N|\boldsymbol{\lambda}_k^{(\Lambda_N)}$ are independent Poisson random variables under $\nu_{N,\alpha}$, we can calculate that for $\varepsilon > 0$

$$\begin{aligned} \mathcal{L}^f(\varepsilon) &= \limsup_{n\to\infty} \frac{1}{|\Lambda_N|} \sum_{k=1}^{\infty} \log \mathbb{E}_{\nu_{N,\alpha}}\left[\mathrm{e}^{\varepsilon\log(k+1)|\Lambda_N|\boldsymbol{\lambda}_k^{(\Lambda_N)}}\right] && (4.1.10) \\ &= \limsup_{N\to\infty} \sum_{k=1}^{\infty} q_{N,k}^{(\mathrm{bc})}\mathrm{e}^{\alpha k}\left(\mathrm{e}^{\varepsilon\log(k+1)} - 1\right). && (4.1.11) \end{aligned}$$

For $\alpha \leq 0$ and $\mathrm{bc} \in \{\varnothing, \mathrm{Dir}\}$ (or $\alpha < 0$ and $\mathrm{bc} = \mathrm{per}$), the sum $\sum_{k=1}^{\infty} q_{N,k}^{(\mathrm{bc})}\mathrm{e}^{\alpha k}$

converges (see Appendix A). If $\alpha < 0$, then $\frac{1}{2}\alpha k + \varepsilon \log(k+1) < 0$ eventually and $\mathcal{L}^f(\varepsilon) < +\infty$. To deal with $\alpha = 0$ and bc $\in \{\varnothing, \mathrm{Dir}\}$, we note that we can bound

$$\mathcal{L}^f(\varepsilon) < \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\frac{d}{2}}} \mathrm{e}^{\varepsilon \log(k+1)} < \frac{2^{\varepsilon}}{(4\pi)^{\frac{d}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\frac{d}{2}-\varepsilon}}. \tag{4.1.12}$$

Therefore for $\varepsilon < \frac{d}{2}$ we have $\mathcal{L}^f(\varepsilon) < +\infty$ and therefore exponential tightness for $\nu_{N,\alpha}$.

$\square$

Lemma 4.1.4. *The limiting logarithmic moment generating function for* $\boldsymbol{\lambda}^{(\Lambda_N)}(\omega_P)$ *exists and is given by*

$$\mathcal{L}(t) = \sum_{k \in \mathbb{N}} q_k \mathrm{e}^{\beta \alpha k} \left(\mathrm{e}^{t_k} - 1\right) < \infty, \qquad t \in \ell_\infty(\mathbb{R}). \tag{4.1.13}$$

Moreover, $\mathcal{L}$ *is Gâteaux differentiable, lower semicontinuous, and strictly convex.*

Proof. First, let us evaluate the logarithmic moment generating function. Recall, that our reference process is a independent superposition of countably many independent marked Poisson point processes. Denote the marginal law of $\boldsymbol{\lambda}_k^{(N)}$ by $\nu_N^{(k)}$, then we have,

$$\begin{aligned} \mathcal{L}(t) &= \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_{\nu_{N,\alpha}} \left[\exp\left(|\Lambda_N| \langle t, \boldsymbol{\lambda}^{(N)} \rangle\right)\right] & (4.1.14) \\ &= \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \sum_{k\in\mathbb{N}} \log \mathbb{E}_{\nu_N^{(k)}} \left[\exp\left(|\Lambda_N| t_k \boldsymbol{\lambda}_k^{(N)}\right)\right] & (4.1.15) \\ &= \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \sum_{k\in\mathbb{N}} |\Lambda_N| \, q_{N,k}^{(\mathrm{bc})} \mathrm{e}^{\alpha k} \left(\mathrm{e}^{t_k} - 1\right) = \sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\alpha k} \left(\mathrm{e}^{t_k} - 1\right). & (4.1.16) \end{aligned}$$

Here, we were able to evaluate the expectation with respect to $\nu_N^{(k)}$ using the logarithmic moment generating function for a Poisson distribution and recalling that $|\Lambda_N| \boldsymbol{\lambda}_k^{(N)}$ is Poisson with mean $|\Lambda_N| q_{N,k}^{(\mathrm{bc})} \mathrm{e}^{\alpha k}$.

To see that $\mathcal{L}(t)$ is finite, note that $t \in \ell_\infty(\mathbb{R})$ implies that $T := \sup_{j\in\mathbb{N}} t_j$ is finite. Hence

$$\mathcal{L}(t) \le \left(\mathrm{e}^{T} - 1\right) \overline{q}^{(\alpha)} < \infty. \tag{4.1.17}$$

To confirm Gâteaux differentiability, let $t, s \in \ell_\infty(\mathbb{R})$ and consider

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{L}(t+\epsilon s) = \sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\alpha k} s_k \mathrm{e}^{t_k+\epsilon s_k}. \tag{4.1.18}$$

This sum is finite because t and s are bounded above and $\overline{q}^{(\alpha)} < +\infty$ for $\alpha \le 0$ ($\alpha < 0$ if bc = per). In particular, the derivative is defined at $\epsilon = 0$, and hence $\mathcal{L}$ is Gâteaux differentiable.

Lower semicontinuity is an immediate consequence of Fatou's Lemma. For any sequence $t^{(n)} \to t$ in $\ell_\infty(\mathbb{R})$,

$$\liminf_{n\to\infty} \mathcal{L}\left(t^{(n)}\right) = \liminf_{n\to\infty} \sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\alpha k}\left(\mathrm{e}^{t_k^{(n)}} - 1\right) \ge \sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\alpha k}\left(\mathrm{e}^{t_k} - 1\right) = \mathcal{L}(t). \tag{4.1.19}$$

To show strict convexity, consider distinct $t, s \in \ell_\infty(\mathbb{R})$ and $\lambda \in (0,1)$. Then

$$\begin{aligned} \mathcal{L}(\lambda s + (1-\lambda)t) &= \sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\alpha k}\left(\mathrm{e}^{\lambda s_k + (1-\lambda)t_k} - 1\right) && (4.1.20)\\ &< \lambda \sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\alpha k}\mathrm{e}^{s_k} + (1-\lambda)\sum_{k\in\mathbb{N}} q_k \mathrm{e}^{\alpha k}\mathrm{e}^{t_k} - \overline{q}^{(\alpha)} && (4.1.21)\\ &= \lambda\mathcal{L}(s) + (1-\lambda)\mathcal{L}(t), && (4.1.22) \end{aligned}$$

where the strict inequality follows from the strict convexity of the exponential function.

□

Remark 4.1.5. *If we do not have $\alpha \le 0$, then we do not have $\mathcal{L}(t) < \infty$ for all $t \in \ell_\infty(\mathbb{R})$. To see this, let t be a constant sequence $t_k = C > 0$. Then $\mathcal{L}(t) = C\overline{q}^{(\alpha)} = \infty$ if $\alpha > 0$.* ◇

Lemma 4.1.6. *For all $x \in \ell_1(\mathbb{R}_+)$, we have*

$$\mathcal{L}^*(x) := \sup_{t\in\ell_\infty(\mathbb{R})} \left\{\langle t, x\rangle - \mathcal{L}(t)\right\} = I_\alpha(x). \tag{4.1.23}$$

PROOF. Define

$$g_x(t) := \langle t, x\rangle - \mathcal{L}(t) = \sum_{k=1}^{\infty} \left[x_k t_k + q_k \mathrm{e}^{\alpha k}\left(1 - \mathrm{e}^{t_k}\right)\right]. \tag{4.1.24}$$

Since g_s os concave, let us search for critical points of g_x. We know g_x is Gâteaux differentiable because it is the sum of a linear term (with coefficients $x \in \ell_1(\mathbb{R}_+)$) and a Gâteaux differentiable term. Taking the Gâteaux derivative of g_x gives us

$$\mathrm{d}g_x(t; s) = \sum_{k=1}^{\infty} s_k \left(x_k - q_k \mathrm{e}^{\alpha k} \mathrm{e}^{t_k}\right), \qquad \forall t, s \in \ell_\infty(\mathbb{R}). \tag{4.1.25}$$

Now t is a critical point if and only if $\mathrm{d}g_x(t; s) = 0\ \forall s \in \ell_\infty(\mathbb{R})$. This means that we want to investigate the sequence $\tilde{t}_k = \log {x_k}/{q_k \mathrm{e}^{\alpha k}}$. If $\tilde{t} \in \ell_\infty(\mathbb{R})$, then this gives us the supremum, and a simple substitution tells us that $\mathcal{L}^*(x) = I(x)$ for such x. Unfortunately, this is not necessarily the case.

Nevertheless, these critical points will give us the supremum over all sequences in $(\mathbb{R} \cup \{-\infty\})^{\mathbb{N}}$. Since $\ell_\infty(\mathbb{R}) \subset (\mathbb{R} \cup \{-\infty\})^{\mathbb{N}}$, we have

$$\mathcal{L}^*(x) = \sup_{t \in \ell_\infty(\mathbb{R})} g_x(t) \leq \sup_{t \in (\mathbb{R} \cup \{-\infty\})^{\mathbb{N}}} g_x(t) = I_\alpha(x). \tag{4.1.26}$$

To find the reverse inequality, let us consider

$$t_k^{(K)} = \begin{cases} \mathbb{1}\{k \leq K\} \log \frac{x_k}{q_k \mathrm{e}^{\alpha k}} & : x_k \neq 0, \\ -K\mathbb{1}\{k \leq K\} & : x_k = 0. \end{cases} \tag{4.1.27}$$

Since $t^{(K)}$ truncates, it is clearly in $\ell_\infty(\mathbb{R})$ for all K. Now let us substitute it into

g_x.

$$g_x\left(t^{(K)}\right) = \sum_{k\le K: x_k\neq 0}\left(x_k\log\frac{x_k}{q_k\mathrm{e}^{\alpha k}} - x_k + q_k\mathrm{e}^{\alpha k}\right) + \sum_{k\le K: x_k=0} q_k\mathrm{e}^{\alpha k}\left(1-\mathrm{e}^{-K}\right) \tag{4.1.28}$$

$$= \sum_{k=1}^{K}\left[x_k\left(\log\frac{x_k}{q_k\mathrm{e}^{\alpha k}} - 1\right) + q_k\mathrm{e}^{\alpha k}\right] - \mathrm{e}^{-K}\sum_{k\le K:\, x_k=0} q_k\mathrm{e}^{\alpha k} \tag{4.1.29}$$

$$\xrightarrow{K\to\infty} I_\alpha\left(x\right). \tag{4.1.30}$$

In the second equality, we have used the convention that $0\log 0 = 0$. The limit is a slight abuse of notation: if the sum defining $I_\alpha\left(x\right)$ converges then this is a true limit, alternatively if $I_\alpha\left(x\right) = +\infty$ then $g_x\left(t^{(K)}\right) \to +\infty$ as $K\to\infty$.

This sequence $\left\{t^{(K)}\right\}_{K\in\mathbb{N}}$ shows that for $x\in\ell_1\left(\mathbb{R}_+\right)$,

$$\mathcal{L}^*\left(x\right) = \sup_{t\in\ell_\infty(\mathbb{R})} g_x\left(t\right) \ge I_\alpha\left(x\right), \tag{4.1.31}$$

as required.

□

Using Baldi's Theorem (Lemma 2.1.14) in conjunction with Lemmas 4.1.3, 4.1.4 and 4.1.6, we conclude with the statement in Proposition 4.1.1.

□

4.2 Cycle Mean Field Model

The Cycle Mean Field (CM) Model is an easy extension beyond the non-interacting case that demonstrates how we would like to proceed in our later models if they did not present certain obstacles.

Theorem 4.2.1 (**Large deviations principle for CM models**). *For any $d\in\mathbb{N}, a>0$, $\alpha\le 0$ and $\mathrm{bc}\in\{\varnothing,\mathrm{Dir}\}$ (or $\alpha<0$ and $\mathrm{bc}=\mathrm{per}$) the sequence $\left\{\nu^{\text{(CM)}}_{N,\alpha}\right\}_{N\ge 1}$ satisfies an LDP on $\ell_1(\mathbb{R}_+)$ with rate $|\Lambda_N|$ and rate function*

$$I^{\text{(CM)}}_\alpha(x) = \beta H^{\text{(CM)}}(x) + I_\alpha(x) - \inf_{y\in\ell_1(\mathbb{R}_+)}\{\beta H^{\text{(CM)}}(y) + I_\alpha(y)\}. \tag{4.2.1}$$

PROOF. We prove this by using Varadhan's Lemma (see Lemmas 2.1.3 and 2.1.4). The tilt $H^{(CM)}$ is clearly bounded below, so we only need to show it is continuous.

Let us write $H^{(CM)} = G \circ \Gamma$, where $G\colon \mathbb{R}_+ \to \mathbb{R}_+$, $x \to \frac{a}{2}x^2$ and $\Gamma : \ell_1(\mathbb{R}_+) \to \mathbb{R}_+$, $x \mapsto \sum_{k\in\mathbb{N}} x_k$. G is clearly continuous. Note that Γ is a bounded linear operator ($|\Gamma(x)| \leq |x|_1$), and is therefore continuous. Hence $H^{(CM)}$ is continuous by virtue of being a composition of two continuous functions.

□

Remark 4.2.2. *The proof of Theorem 4.2.1 follows easily from Varadhan's Lemma because the function $\Gamma : \ell_1(\mathbb{R}_+) \to \mathbb{R}_+$ is continuous. In the particle mean field models the function $D : \ell_1(\mathbb{R}_+) \to [0, +\infty]$ plays a similar role, but we do not have continuity. It is this difficulty that we are trying to overcome in Section 4.3.* ◇

4.3 Particle Mean Field Models

We first prove the generalised particle mean field (GPM) model large deviation principle in Theorem 4.3.1. The specific physical particle mean field (PM) model is then a corollary of this - see Corollary 4.3.11. For this section, we denote $\mathcal{I}(x) = \mathcal{I}_0(x)$.

Theorem 4.3.1 (Large deviation principle for GPM models). *If G is continuous and bounded below, then the associated sequence of GPM measures $\{\nu_N^{(GPM)}\}_{N\geq1}$ satisfies an LDP on $\ell_1(\mathbb{R}_+)$ with rate $|\Lambda_N|$ and rate function*

$$I^{(GPM)}(x) = \overline{G} \circ D(x) + I(x) - \inf_{y\in\ell_1(\mathbb{R}_+)} \{\overline{G} \circ D(y) + I(y)\}, \tag{4.3.1}$$

with

$$\overline{G}(D) = \inf_{D'\geq D} G(D'). \tag{4.3.2}$$

PROOF. To prove the large deviation principle for $\nu_N^{(GPM)}$ one would hope to use Varadhan's Lemma. However, the particle density D is only lower semicontinuous - we cannot generally say anything about the composition $G \circ D$. Using [GZ93], one would arrive at lower and upper bounds for $\mathbb{E}_{\nu_N}[\mathrm{e}^{-|\Lambda_N|G\circ D}]$ using the upper and the lower semicontinuous regularisation of $G \circ D$, respectively. Unfortunately, this can be far too crude: in the physical PM model with convex quadratic G, the upper semicontinuous regularisation of the $G \circ D$ equals infinity and does not provide a helpful lower bound for the large deviation principle. Our strategy is therefore twofold. For the large deviation upper bound we use the lower semicontinuous regularisation in conjunction with the corresponding bound in Varadhan's Lemma. We obtain the corresponding large deviation lower bound by conditioning that the empirical cycle count is supported on a finite-dimensional subspace. On this event we can replace our measure by the corresponding measure with finite dimensional mark space. On this subspace D is in fact continuous and the upper semicontinuity of G allows us to use Varadhan's Lemma to get a lower bound. To remove the cutoff parameter we will construct finite-dimensional sequences approximating the infimum of the corresponding lower bound. We start with the following technical lemma. Part 2) of Lemma 4.3.2 is in fact a generalisation of [Moo99, Proposition 3.45], which is already sufficient for our purposes here. Nevertheless, when we used this result in Lemma 3.1.12 we did require the greater generality.

Lemma 4.3.2. *Let (X, τ_X) be a general topological space, (Y, τ_Y) be a topological space where Y has been equipped with a total ordering and τ_Y is the order topology, and Z be a totally ordered space. Let $f : Y \to Z$ be lower semicontinuous with respect to τ_Y and $g : X \to Y$ be lower semicontinuous with respect to τ_X. Then*

1. *$F : Y \to Z$ is lower semicontinuous with respect to τ_Y and non-decreasing, where*
$$F(y) = \inf_{y' \geq y} f(y'), \tag{4.3.3}$$

2. *$h = F \circ g$ is lower semicontinuous with respect to τ_X.*

Proof. Firstly F is non-decreasing by being an infimum of a function of ever smaller sets as y increases. For the lower semicontinuity of F, begin by recalling that the lower semicontinuity of f tells us that $f^{-1}(\{z : z > c\})$ is open (it is in τ_Y) and the complement $f^{-1}(\{z : z \leq c\})$ is closed. Then since τ_Y is the order topology, $f^{-1}(\{z : z \leq c\})$ is a intersection of closed "double rays" of the form

$\{y : y \leq a \text{ or } y \geq b\}$, and closed single rays of the form $\{y : y \leq a\}$ and $\{y : y \geq b\}$. From the definition of F, we know that $y \in f^{-1}(\{z : z \leq c\})$ and $y' \leq y$ implies that $y' \in F^{-1}(\{z : z \leq c\})$. Therefore we can construct $F^{-1}(\{z : z \leq c\})$ by taking the above closed-set construction of $f^{-1}(\{z : z \leq c\})$ and replacing the double rays and the right-hand rays $\{y : y \geq b\}$ with the whole set Y. This construction is an intersection of closed sets and shows that $F^{-1}(\{z : z \leq c\})$ is closed whilst $F^{-1}(\{z : z > c\})$ is open.

We now address the semicontinuity. To begin with, note that $h^{-1}(\{z : z > c\}) = g^{-1} \circ F^{-1}(\{z : z > c\})$. Now since F is non-decreasing, $F^{-1}(\{z : z > c\}) = \{y : y \geq d\}$ or $F^{-1}(\{z : z > c\}) = \{y : y > d\}$ for some $d = d(c)$. The lower semicontinuity of F allows us to restrict this further to $F^{-1}(\{z : z > c\}) = \{y : y > d\}$ because this preimage is open in the order topology. Finally, the lower semicontinuity of g with respect to τ_X tells us that $g^{-1}(\{y : y > d\}) \in \tau_X$.

□

Proposition 4.3.3 (Upper bound GPM-model). *In addition to the hypotheses of Lemma 4.3.2, let f be bounded below and $\{\mathbb{P}_N\}_{N \in \mathbb{N}}$ be a sequence of probability measures on the regular topological space (X, τ_X) satisfying the LDP with a good rate function $\mathcal{I} : X \to [0, +\infty]$ and rate r_N. Then*

$$\limsup_{N \to \infty} \frac{1}{r_N} \log \mathbb{E}_N \left[\exp\left(-r_N f \circ g\right)\right] \leq - \inf_X \left\{\mathcal{I} + F \circ g\right\}. \tag{4.3.4}$$

In particular,

$$\limsup_{N \to \infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_{\nu_N} \left[\exp\left(- |\Lambda_N| \, G \circ D\right)\right] \leq - \inf_X \left\{I + \overline{G} \circ D\right\}. \tag{4.3.5}$$

Proof. The statement (4.3.4) follows easily from Lemma 2.1.4 using the inequality $F \leq f$ everywhere, the lower semicontinuity of F, and the fact that F is bounded below if and only if f is. Then (4.3.5) follows from (4.3.4) and the lower semicontinuity of G.

□

Remark 4.3.4. *The bound* (4.3.5) *is the best that this type of argument can do* *because $\overline{G} \circ D$ is the lower semicontinuous regularisation (the greatest lower semicontinuous minorant) of $G \circ D$, as proven in the following Lemma.* $\diamond$

Lemma 4.3.5. *Let $G : [0, +\infty] \to [-\infty, +\infty]$ be lower semicontinuous. Then $\overline{G} \circ D : \ell_1(\mathbb{R}_+) \to [-\infty, +\infty]$ is the lower semicontinuous regularisation of $G \circ D$.*

Proof. From Lemma 4.3.2, we know that $\overline{G} \circ D$ is lower semicontinuous, and by the construction of $\overline{G}$ we know that $\overline{G} \circ D \leq G \circ D$ everywhere.

Suppose for contradiction that $h : \ell_1(\mathbb{R}_+) \to [-\infty, +\infty]$ is lower semicontinuous, that $h \leq G \circ D$ everywhere, and that there exists $x \in \ell_1(\mathbb{R}_+)$ such that $h(x) > \overline{G} \circ D(x)$. From the construction of $\overline{G}$, we know that there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\limsup_{n\to\infty} G(D(x) + A_n) = \overline{G} \circ D(x)$, upon which we can impose $A_n \ll n$ if necessary. Now consider the perturbation $\epsilon^{(n)} \in \ell_1(\mathbb{R}_+)$, where $\epsilon^{(n)}_k = \frac{A_n}{n} \mathbb{1}\{n = k\}$ for all $k, n \in \mathbb{N}$, so $\lim_{n\to\infty} |\epsilon^{(n)}|_1 = 0$. The perturbed particle density $D(x + \epsilon^{(n)}) = D(x) + A_n$. Therefore

$$h(x) > \overline{G} \circ D(x) = \limsup_{n\to\infty} G \circ D(x + \epsilon^{(n)}) \geq \liminf_{n\to\infty} h(x + \epsilon^{(n)}), \tag{4.3.6}$$

contradicting the lower semicontinuity of h. This proves $\overline{G} \circ D$ is the greatest lower semicontinuous minorant of $G \circ D$.

$\square$

Now we prove the lower bound estimate in less generality: we consider the functions G and D directly.

Proposition 4.3.6 (Lower bound GPM-model).

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_{\nu_N}\left[\exp\left(-|\Lambda_N| G \circ D\right)\right] \geq - \inf_{\ell_1(\mathbb{R}_+)} \left\{I + \overline{G} \circ D\right\}. \tag{4.3.7}$$

Proof. The strategy for proving the lower bound is to first introduce a cut-off parameter as done in [ACK11] and Chapter 3, that is, we change the measure to obtain a finite-dimensional problem which gives continuity to $G \circ D$ and thus a large deviation lower bound for the finite-dimensional space. The final step is then to remove the cut-off parameter. As G is not necessarily non-decreasing, removing the cut-off is not as straightforward as in [ACK11] and our previous discussion. We thus need to construct a sequence for the finite dimensional spaces which allows for energy estimates and at the same time gives convergence towards the lower semicontinuous regularisation.

Step 1: Restriction of the mark space. We will approximate the mark space E by the cut-off version

$$E^{(K)} = \bigcup_{k=1}^{K} \mathcal{C}_k, \quad K \in \mathbb{N}. \tag{4.3.8}$$

The Poisson reference process on $\mathbb{R}^d \times E^{(K)}$ is denoted by $\mathsf{Q}_\alpha^{(K)}$, and the corresponding measure on $\mathbb{R}_+^K$ which is isomorphic to $\pi_K(\ell_1(\mathbb{R}_+))$ with $\pi_K\colon \ell_1(\mathbb{R}_+) \to \mathbb{R}_+^K, x \mapsto (x_1, \ldots, x_K)$, is denoted $\nu_N^{(K)} = \mathsf{Q}_\alpha^{(K)} \circ \left(\boldsymbol{\lambda}^{(\Lambda_N)}\right)^{-1} = \nu_N \circ \pi_K^{-1}$. We obtain a large deviation principle for the cut-off version in the following.

Lemma 4.3.7. *For given $K \in \mathbb{N}$, the sequence $(\nu_N^{(K)})_{N\in\mathbb{N}}$ satisfies an LDP on $\mathbb{R}_+^K$ with rate $|\Lambda_N|$ and rate function*

$$I^{(K)}(x) = \sum_{k=1}^{K} \left(x_k \log \frac{x_k}{q_k} - x_k + q_k \right), \qquad \forall x \in \mathbb{R}_+^K. \tag{4.3.9}$$

Proof. Since the projection π_K is continuous, we can apply the contraction principle (Lemma 2.1.15) to obtain a variational form of the rate function

$$I^{(K)}(x) = \inf_{\widetilde{x}\in\ell_1(\mathbb{R}_+)\colon \pi_K(\widetilde{x})=x} I(\widetilde{x}), \tag{4.3.10}$$

where I is the rate function for ν_N in Proposition 4.1.1.

For $\widetilde{x} \in \ell_1(\mathbb{R}_+)$ with $\pi_K(\widetilde{x}) = x \in \mathbb{R}_+^K$,

$$I(\widetilde{x}) = \sum_{k=1}^{K} x_k \left(\log \frac{x_k}{q_k} - 1 \right) + \sum_{k=K+1}^{\infty} \widetilde{x}_k \left(\log \frac{\widetilde{x}_k}{q_k} - 1 \right) + \overline{q}. \tag{4.3.11}$$

Thus it suffices to minimise the second term, which can be done term-wise. The infimum is given by $\widetilde{x} = y$, where

$$y_k = \begin{cases} x_k, & k = 1, \dots, K \\ q_k, & k > K. \end{cases} \tag{4.3.12}$$

This gives us

$$I^{(K)}(x) = I(y) = \sum_{k=1}^{K} x_k \left(\log \frac{x_k}{q_k} - 1 \right) - \sum_{j=K+1}^{\infty} q_k + \overline{q}, \tag{4.3.13}$$

as required.

□

Step 2: Lower bound. We begin by inserting an indicator the lower bound,

$$\mathbb{E}_{\nu_N} \left[\exp\left(-\beta |\Lambda_N| G \circ D \right) \right] \geq \mathbb{E}_{\nu_N} \left[\exp\left(-\beta |\Lambda_N| G \circ D \right) \mathbb{1}\{ \boldsymbol{\lambda}^{(N)} \in \mathbb{R}_+^K \} \right], \tag{4.3.14}$$

where we identified $\mathbb{R}_+^K$ with the corresponding subspace in $\ell_1(\mathbb{R}_+)$. On that event we can replace D by $D^{(K)}$, where

$$D^{(K)}(x) = \sum_{j=1}^{K} j x_j. \tag{4.3.15}$$

We now want to replace $\mathbb{E}_{\nu_N}$ by $\mathbb{E}_{\nu_N^{(K)}}$. These measures are not actually equal on the event $\left\{ \boldsymbol{\lambda}^{(N)} \in \mathbb{R}_+^K \right\}$ but are proportional to each other, with the factor being $\nu_N \left(\boldsymbol{\lambda}^{(N)} \in \mathbb{R}_+^K \right)$.

Lemma 4.3.8.

$$\frac{1}{|\Lambda_N|} \log \nu_N \left(\boldsymbol{\lambda}^{(N)} \in \mathbb{R}_+^K \right) = - \sum_{k=K+1}^{\infty} q_k. \tag{4.3.16}$$

PROOF. Recall that $\boldsymbol{\lambda}_k^{(N)}(\omega_P)$ are Poisson random variables with mean $|\Lambda_N| q_k$. Therefore

$$\nu_N \left(\boldsymbol{\lambda}_k^{(N)}(\omega_P) = 0 \right) = \mathrm{e}^{-|\Lambda_N| q_k}. \tag{4.3.17}$$

The independence of $\boldsymbol{\lambda}_k^{(N)}(\omega_P)$ under ν_N then gives the result.

□

Since $D^{(K)}$ is continuous, the upper semicontinuity of G implies that the finite dimensional approximation $G \circ D^{(K)}$ is also upper semicontinuous. We then obtain a large deviation lower bound using Lemma 4.3.7 and Varadhan's Lemma,

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_{\nu_N} \left[\exp\left(-\beta|\Lambda_N| G\circ D\right)\right] \geq -\inf_{\mathbb{R}_+^K} \{I^{(K)} + G\circ D^{(K)}\} - \sum_{k=K+1}^{\infty} q_k. \tag{4.3.18}$$

Step 3: Removing the cut-off parameter. We are left to remove the cut-off by taking $K \to \infty$ and to prove that in this limit $G \circ D^{(K)}$ is replaced by $\overline{G} \circ D$. The sum $\sum_{k=K+1}^{\infty} q_k$ vanishes in this limit because $q \in \ell_1$.

Lemma 4.3.9.

$$\limsup_{K\to\infty} \inf_{\mathbb{R}_+^K} \{I^{(K)} + G\circ D^{(K)}\} \leq \inf_{\ell_1(\mathbb{R}_+)} \left\{I + \overline{G}\circ D\right\}. \tag{4.3.19}$$

Proof. Fix $\widetilde{x} \in \ell_1(\mathbb{R}_+)$ and, for $K \in \mathbb{N}$, consider $x^K = \pi_K(\widetilde{x})$. By (4.3.9), we have $I^{(K)}(x^K) \leq I(\widetilde{x})$. Suppose $D(\widetilde{x}) = +\infty$ so that $D^{(K)}\left(x^K\right) \to \infty$ as $K \to \infty$. Then note $\overline{G}(+\infty) = G(+\infty)$, so that the upper semicontinuity of G gives us $\limsup_{K\to\infty} G\circ D^{(K)}\left(x^K\right) \leq G(+\infty) = \overline{G}\circ D(\widetilde{x})$.

Now instead suppose $D(\widetilde{x}) < +\infty$. From the construction of $\overline{G}$, we know that there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\limsup_{n\to\infty} G(D(x) + A_n) = \overline{G}\circ D(x)$. We replace x^K with $\widehat{x}^K = x^K + \epsilon^{(K)} \in \ell_1(\mathbb{R}_+)$ where

$$\epsilon_k^{(K)} = \frac{1}{K}\left(D(\widetilde{x}) - D(x^{(K)}) + A_K\right) \mathbb{1}\{K = k\}, \qquad \forall k, K \in \mathbb{N}. \tag{4.3.20}$$

By construction, $\limsup_{K\to\infty} G\circ D\left(\widehat{x}^K\right) = \overline{G}\circ D(x)$. Also

$$\begin{aligned} I^{(K)}(\widehat{x}^K) =& I^{(K)}(x^K) + \left(I^{(K)}(\widehat{x}^K) - I^{(K)}(x^K)\right) && (4.3.21) \\ \leq& I(\widetilde{x}) - \epsilon_K^{(K)} - \epsilon_K^{(K)} \log q_K + \left[\left(x_K^K + \epsilon_K^{(K)}\right)\log\left(x_K^K + \epsilon_K^{(K)}\right) - x_K^K \log x_K^K\right]. && (4.3.22) \end{aligned}$$

Since the map $x \mapsto x\log x$ is continuous on $\mathbb{R}_+$ (recall $0\log 0 = 0$), the last term vanishes in the $K \to \infty$ limit if $\epsilon_K^{(K)} \ll 1$. If $\epsilon_K^{(K)} \ll 1/\log K$, then all excess terms vanish and we have $\limsup_{K\to\infty} I^{(K)}(\widehat{x}^K) \leq I(\widetilde{x})$. Since $D\left(\widetilde{x}\right) - D\left(x^{(K)}\right) \to 0$, this only requires $A_K \ll {}^{K}/_{\log K}$ - which we are free to impose.

In both the case $D\left(\widetilde{x}\right) = +\infty$ and the case $D\left(\widetilde{x}\right) < +\infty$ we have that

$$\limsup_{K\to\infty} \left(I^{(K)} + G \circ D^{(K)}\right)\left(\widehat{x}^K\right) \leq \left(I + \overline{G} \circ D\right)(\tilde{x}) \tag{4.3.23}$$

as required.

□

This completes the proof of Lemma 4.3.6.

□

We finally combine Proposition 4.3.3 and Proposition 4.3.6 to finish the proof for Theorem 4.3.1.

□

To use Theorem 4.3.1 to get the corresponding result for the physical case (Corollary 4.3.11), we now only need to find $\overline{G} \circ D$ for this specific case.

Lemma 4.3.10. *For all $\mu \in \mathbb{R}$, the lower semicontinuous regularisation of $H^{(PM)}_{\mu}$ is given as*

$$\begin{aligned} H^{(PM)}_{\mu,\text{l.s.c.}}(x) &= H^{(PM)}_{\mu}(x) - \frac{1}{2a}\left(\mu - aD(x)\right)_+^2 \\ &= \begin{cases} -\mu D(x) + \frac{a}{2}D(x)^2 & : D(x) \geq \frac{\mu}{a}, \\ -\frac{\mu^2}{2a} & : D(x) < \frac{\mu}{a}, \end{cases} \qquad x \in \ell_1(\mathbb{R}_+). \end{aligned} \tag{4.3.24}$$

In particular, for all $\mu \leq 0$, $H^{(PM)}_{\mu,\text{l.s.c.}} \equiv H^{(PM)}_{\mu}$.

Proof. This follows from Lemma 4.3.5 with $G : x \mapsto -\mu x + \frac{a}{2}x^2$.

□

Corollary 4.3.11 (**Large deviation principle for PM models**). *For any $d \in \mathbb{N}$, $a > 0$, $\mu \in \mathbb{R}$, and $\alpha \leq 0$ and* bc $\in \{\varnothing, \text{Dir}\}$ *(or $\alpha < 0$ and* bc = per*) the following holds. The sequence $\left\{\nu^{(PM)}_{N,\mu,\alpha}\right\}_{N\geq 1}$ satisfies an LDP on $\ell_1(\mathbb{R}_+)$ with rate $|\Lambda_N|$ and*

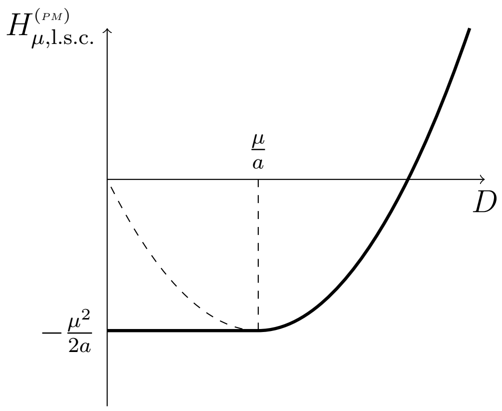


Figure 4.1: Sketch of $H^{(PM)}_{\mu,\text{l.s.c.}}$ as a function of the total particle density D.

rate function

$$I^{(PM)}_{\mu,\alpha}(x) = \beta H^{(PM)}_{\mu+\alpha,\text{l.s.c.}}(x) + I_0(x) - \inf_{y\in\ell_1(\mathbb{R}_+)}\{\beta H^{(PM)}_{\mu+\alpha,\text{l.s.c.}}(y) + I_0(y)\}, \tag{4.3.25}$$

with

$$H^{(PM)}_{\mu,\text{l.s.c.}}(x) = H^{(PM)}_{\mu}(x) - \frac{1}{2a}\left(\mu - aD(x)\right)_+^2 = \begin{cases} -\mu D(x) + \frac{a}{2}D(x)^2 & : D(x) \geq \frac{\mu}{a}, \\ -\frac{\mu^2}{2a} & : D(x) < \frac{\mu}{a}. \end{cases} \tag{4.3.26}$$

Remark 4.3.12. *For $\mu + \alpha \leq 0$, the rate function in Theorem 4.3.11 reads*

$$I^{(PM)}_{\mu,\alpha}(x) = \beta H^{(PM)}_{\mu+\alpha}(x) + I(x) - \inf_{y\in\ell_1(\mathbb{R}_+)}\{\beta H^{(PM)}_{\mu+\alpha}(y) + I(y)\}. \tag{4.3.27}$$

$\diamond$

4.4 Full Cycle HYL Model

Theorem 4.4.1 (Large deviations principle for FCH model). *For any $d \in \mathbb{N}$, $a > b \geq 0$, $\mu \in \mathbb{R}$, $\alpha \leq 0$ and $\text{bc} \in \{\varnothing, \text{Dir}\}$ (or $\alpha < 0$ and $\text{bc} = \text{per}$) the following holds. The sequence $\left\{\nu^{(FCH)}_{N,\mu,\alpha}\right\}_{N\in\mathbb{N}}$ satisfies an LDP on $\ell_1(\mathbb{R}_+)$ with rate $|\Lambda_N|$ and rate function*

$$I^{(FCH)}_{\mu,\alpha}(x) = \beta H^{(FCH)}_{\mu+\alpha,\text{l.s.c.}}(x) + I_0(x) - \inf_{y\in\ell_1(\mathbb{R})}\{\beta H^{(FCH)}_{\mu+\alpha,\text{l.s.c.}}(y) + I_0(y)\}, \tag{4.4.1}$$

with

$$H^{(FCH)}_{\mu,\mathrm{l.s.c.}}(x) = H^{(FCH)}(x)_\mu - \frac{1}{2(a-b)}\left(\mu - aD(x)\right)_+^2 \tag{4.4.2}$$

$$= -\frac{b}{2}\sum_{k=1}^{\infty} k^2 x_k^2 + \begin{cases} -\mu D(x) + \frac{a}{2}D(x)^2 & : D(x) \geq \frac{\mu}{a}, \\ -\frac{b}{a-b}\left(-\mu D(x) + \frac{a}{2}D(x)^2\right) - \frac{\mu^2}{2(a-b)} & : D(x) < \frac{\mu}{a}. \end{cases} \tag{4.4.3}$$

This section proves Theorem 4.4.1 by building upon the results of Section 4.3. We rewrite the Hamiltonian in two equivalent ways,

$$H^{(FCH)}_{\mu}(x) = -\mu D(x) + \frac{(a-b)}{2} D(x)^2 + \frac{b}{2}\sum_{\substack{j,k=1 \\ j\neq k}}^{\infty} jk x_j x_k \tag{4.4.4}$$

$$= -\mu D(x) + \frac{a}{2} D(x)^2 - \frac{b}{2}\sum_{k=1}^{\infty} k^2 x_k^2. \tag{4.4.5}$$

Note that that the right hand side in (4.4.4) is the sum of a PM Hamiltonian with interaction strength $(a-b)$ and a lower semicontinuous and non-negative term. On the other hand, (4.4.5) expresses $H^{(FCH)}$ as the sum of a PM Hamiltonian, and an upper semicontinuous and non-positive term. Let us introduce the following notations

$$H^{(PM)}_{\mu,a}(x) = -\mu D(x) + \frac{a}{2} D(x)^2, \quad a > 0, \tag{4.4.6}$$

$$H_+(x) = \frac{b}{2}\sum_{\substack{j,k=1 \\ j\neq k}}^{\infty} jk x_j x_k, \qquad H_-(x) = -\frac{b}{2}\sum_{k=1}^{\infty} k^2 x_k^2. \tag{4.4.7}$$

Thus $H^{(FCH)}_{\mu} = H^{(PM)}_{\mu,a} + H_- = H^{(PM)}_{\mu,a-b} + H_+$.

Lemma 4.4.2. *For $b > 0$, H_- is upper semicontinuous and H_+ is lower semicontinuous on $\ell_1(\mathbb{R}_+)$.*

Proof. We shall show that $\sum_{\substack{j,k=1 \\ j\neq k}}^{\infty} jk x_j x_k$ and $\sum_{k=1}^{\infty} k^2 x_k^2$ are both lower semicontinuous on $\ell_1(\mathbb{R}_+)$. Suppose $x^{(n)} \to x$ in $\ell_1(\mathbb{R}_+)$. Clearly,

$$\left|x_k^{(n)} - x_k\right| \leq \|x^{(n)} - x\|_{\ell_1} \quad \text{for all } k \in \mathbb{N}. \tag{4.4.8}$$

Furthermore, due to the ℓ_1-convergence and $\ell_1 \subset \ell_\infty$, the term $\left|x_k^{(n)} - x_k\right|$ is bounded in both k and n. Hence

$$\left|(x_k^{(n)})^2 - x_k^2\right| = \left|x_k^{(n)} - x_k\right|\left|x_k^{(n)} + x_k\right| \to 0 \qquad \text{as } n \to \infty. \tag{4.4.9}$$

Applying Fatou's Lemma here proves that H_- is upper semicontinuous. Similarly, for all $(j,k) \in \mathbb{N}^2$,

$$\left|x_j^{(n)} x_k^{(k)} - x_j x_k\right| \leq \left|x_j^{(n)}\right|\left|x_k^{(n)} - x_k\right| + |x_k|\left|x_j^{(n)} - x_j\right| \to 0 \qquad \text{as } n \to \infty. \tag{4.4.10}$$

This convergence in conjunction with Fatou's Lemma shows that H_+ is lower semicontinuous.

$\square$

Lemma 4.4.3. *For $a > b$, the $\ell_1(\mathbb{R}_+)$ lower semicontinuous regularisation of $H_\mu^{(FCH)}$ is given by*

$$H_{\mu,\mathrm{l.s.c.}}^{(FCH)}(x) = H_\mu^{(FCH)}(x) - \frac{(\mu - aD(x))_+^2}{2(a-b)}, \quad x \in \ell_1(\mathbb{R}_+). \tag{4.4.11}$$

Proof. Denote the right hand side of (4.4.11) by h. Clearly, $h(x) \leq H_\mu^{(FCH)}(x)$ and $H_{\mu,(a-b),\mathrm{l.s.c.}}^{(PM)}(x) \leq h(x) = H_{\mu,(a-b),\mathrm{l.s.c.}}^{(PM)}(x) + H_+(x) \leq H_\mu^{(FCH)}(x)$ for all $x \in \ell_1(\mathbb{R}_+)$. We need to show that h is the greatest lower semicontinuous function less or equal to $H_\mu^{(FCH)}$.

Suppose that $x \in \ell_1(\mathbb{R}_+)$ with $D(x) = \infty$. Then since $H_{\mu,(a-b),\mathrm{l.s.c.}}^{(PM)}(x) = \infty$, we have $h(x) = \infty$. Suppose now that $x \in \ell_1(\mathbb{R}_+)$ with $D(x) < \infty$. For any sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x$ as $n \to \infty$ there exists $(\varepsilon^{(n)})_{n\in\mathbb{N}} \subset \ell_1(\mathbb{R})$ such that $x_n = x + \varepsilon^{(n)}$, $\varepsilon^{(n)} \to 0$ as $n \to \infty$, and $x + \varepsilon^{(n)} \in \ell_1(\mathbb{R}_+)$. Furthermore, $\liminf_{n\to\infty}(D(x^{(n)}) - D(x)) \geq 0$ and thus $\liminf_{n\to\infty} D(\varepsilon^{(n)}) \geq 0$. We show that h

is lower semicontinuous by proving that

$$\begin{aligned}
&\liminf_{n\to\infty} \left(h(x^{(n)}) - h(x)\right) \\
&\quad = \liminf_{n\to\infty} \Big\{ -\mu \left(D(x^{(n)}) - D(x)\right) + \frac{a}{2}\left(D(x^{(n)})^2 - D(x)^2\right) \\
&\quad -\frac{b}{2}\left(\sum_{k=1}^{\infty} k^2 \left((x_k^{(n)})^2 - x_k^2\right)\right) + \frac{1}{2(a-b)}\left((\mu - aD(x))_+^2 - (\mu - aD(x^{(n)}))_+^2\right)\Big\} \\
&\quad \geq 0. \qquad (4.4.12)
\end{aligned}$$

If $D\left(\varepsilon^{(n)}\right) \to +\infty$, then $\liminf_{n\to\infty} h\left(x^{(n)}\right) = +\infty$, so we suppose that $D\left(\varepsilon^{(n)}\right)$ is finite and bounded.

We write $\varepsilon_k^{(n)} = \varepsilon_k^{+(n)} - \varepsilon_k^{-(n)}$ with $\varepsilon_k^{+(n)}, \varepsilon_k^{-(n)} \geq 0$. Clearly, $\varepsilon_k^{-(n)} \leq x_k$ for all $k \in \mathbb{N}$. We shall show that

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} k^2 x_k \varepsilon_k^{(n)} = 0, \qquad (4.4.13)$$

and that

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} k^2 (x_k + \varepsilon_k^{(n)})^2 \leq \sum_{k=1}^{\infty} k^2 x_k^2 + \limsup_{n\to\infty} \left(D(\varepsilon^{(n)})\right)^2. \qquad (4.4.14)$$

As $D(x) < \infty$, for all $C > 0$ there exits $K_C \in \mathbb{N}$ such that $kx_k < C$ for all $k > K_C$. Therefore we have

$$\sum_{k=K_C+1}^{\infty} k^2 x_k \varepsilon_k^{(n)} < C \sum_{k=K_C+1}^{\infty} k\varepsilon_k^{(n)} \leq CD\left(\varepsilon^{(n)}\right). \qquad (4.4.15)$$

Then since $\lim_{n\to\infty} \sum_{k=1}^{K_C} k^2 x_k \varepsilon_k^{(n)} = 0$, and $D\left(\varepsilon^{(n)}\right)$ is bounded, we can choose C arbitrarily small to get (4.4.13). To obtain (4.4.14) we just expand

$$\sum_{k=1}^{\infty} k^2 (x_k + \varepsilon_k^{(n)})^2 \leq \sum_{k=1}^{\infty} k^2 x_k^2 + 2\sum_{k=1}^{\infty} k^2 (x_k - \varepsilon_k^{-(n)})\varepsilon_k^{+(n)} + \left(D(\varepsilon^{+(n)})\right)^2. \qquad (4.4.16)$$

The middle term vanishes due to (4.4.13). To show that

$$\limsup_{n\to\infty} \left(D(\varepsilon^{+(n)})\right)^2 \le \limsup_{n\to\infty} \left(D(\varepsilon^{(n)})\right)^2, \tag{4.4.17}$$

note that $D(x) < \infty$ implies that

$$D(\varepsilon^{(n)}) - D(\varepsilon^{+(n)}) = D(\varepsilon^{-(n)}) \le D(x) < \infty. \tag{4.4.18}$$

Hence, for any δ there exists $K \in \mathbb{N}$ such that

$$\sum_{k=K+1}^{\infty} k\varepsilon_k^{-(n)} \le \sum_{k=K+1}^{\infty} kx_k < \frac{\delta}{2}. \tag{4.4.19}$$

On the other hand, there exists for this δ a $n(K) \in \mathbb{N}$ such that

$$\sum_{k=1}^{K} k\varepsilon_k^{-(n)} < \frac{\delta}{2}, \qquad \text{for all } n \ge n(K), \tag{4.4.20}$$

thus showing (4.4.14). We continue with

$$\begin{aligned}\text{r.h.s. of (4.4.12)} \ge \liminf_{n\to\infty} \Bigg\{ &\frac{1}{2(a-b)} \left((\mu - aD(x))_+^2 - (\mu - aD(x^{(n)}))_+^2 \right) \\ &- (\mu - aD(x))\, D(\varepsilon^{(n)}) + \frac{a-b}{2} \left(D(\varepsilon^{(n)})\right)^2 \Bigg\}. \end{aligned} \tag{4.4.21}$$

Recall that $\liminf_{n\to\infty} D(\varepsilon^{(n)}) \ge 0$, and thus we know that $(\mu - aD(x)) < 0$ implies that eventually $(\mu - aD(x) - aD(\varepsilon^{(n)})) < 0$ and $(\mu - aD(x) - aD(\varepsilon^{(n)}) + bD(\varepsilon^{(n)})) < 0$. Suppose that $\mu/a < D(x)$. Then

$$\text{r.h.s. of (4.4.21)} = \liminf_{n\to\infty} \left\{ - (\mu - aD(x))\, D(\varepsilon^{(n)}) + \frac{a-b}{2} D(\varepsilon^{(n)})^2 \right\} \ge 0. \tag{4.4.22}$$

Suppose $\mu/a \ge D(x)$ and $\mu - aD(x) - aD(\varepsilon^{(n)}) \le 0$. Then

$$\text{r.h.s. of (4.4.21)} \ge \frac{1}{2(a-b)} \liminf_{n\to\infty} \left\{ \left(\mu - aD(x) - aD(\varepsilon^{(n)}) + bD(\varepsilon^{(n)})\right)^2 \right\} \ge 0, \tag{4.4.23}$$

and likewise for $\mu/a \geq D(x)$ and $\mu - aD(x) - aD(\varepsilon^{(n)}) > 0$,

$$\text{r.h.s. of (4.4.21)} \geq \frac{1}{2(a-b)} \liminf_{n\to\infty} \Big\{ \big(\mu - aD(x) - aD(\varepsilon^{(n)}) + bD(\varepsilon^{(n)})\big)^2 - \big(\mu - aD(x) - aD(\varepsilon^{(n)})\big)^2 \Big\} \geq 0. \quad (4.4.24)$$

We have established (4.4.12) and thus the lower semicontinuity of h. We finally show that h is the largest lower semicontinuous function less or equal to $H_\mu^{(FCH)}$. Using the lower semicontinuity of h and $h \leq H_\mu^{(FCH)}$, we know that

$$\liminf_{n\to\infty} H_\mu^{(FCH)}(x^{(n)}) \geq h(x) \quad (4.4.25)$$

for any sequence $x^{(n)}$ with $x^{(n)} \to x$ as $n \to \infty$. We pick now a particular sequence $x^{(n)} = x + \varepsilon^{(n)}$ with

$$\varepsilon_k^{(n)} = 1\!\!1\{k = n\} \frac{(\mu - aD(x))_+}{n(a-b)} \quad (4.4.26)$$

to find that

$$h(x) \leq H^{(FCH)}(x) - \frac{(\mu - aD(x))_+}{2(a-b)}. \quad (4.4.27)$$

□

Proposition 4.4.4 (Upper bound FCH-model). *For all $\mu \in \mathbb{R}, \alpha \leq 0$, and $a > b \geq 0$,*

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_{\nu_{N,\alpha}} \left[\mathrm{e}^{-\beta|\Lambda_N| H_\mu^{(FCH)}} \right] \leq - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I(x) + \beta H_{\mu+\alpha,\text{l.s.c.}}^{(FCH)}(x) \right\}. \quad (4.4.28)$$

Proof. The statement follows easily with the upper bound estimate in Varadhan's Lemma (see Lemma 2.1.4) using the inequality $H_\mu^{(FCH)}(x) \geq H_{\mu,\text{l.s.c.}}^{(FCH)}(x) \geq H_{\mu,(a-b),\text{l.s.c.}}^{(PM)}(x)$ for all $x \in \ell_1(\mathbb{R}_+)$, the lower semicontinuity of $H_{\mu,\text{l.s.c.}}^{(FCH)}$, and the fact that $H_{\mu,\text{l.s.c.}}^{(FCH)}(x) \geq \frac{-\mu^2}{2(a-b)}$. Since the energy is bounded below, it satisfies the tail-condition (2.1.6).

□

For the lower bound we are using the lower bound (4.3.7) for the PM model and $H_\mu^{(FCH)} = H_{\mu,a}^{(PM)} + H_-$ with H_- being upper semicontinuous.

Proposition 4.4.5 (Lower bound FCH-model). *For all $\mu \in \mathbb{R}, \alpha \le 0$, and $a > b \ge 0$,*

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_{\nu_{N,\alpha}} \left[\mathrm{e}^{-\beta|\Lambda_N| H^{(\mathrm{FCH})}_{\mu}} \right] \ge - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I(x) + \beta H^{(\mathrm{FCH})}_{\mu+\alpha,\mathrm{l.s.c.}}(x) \right\}. \quad (4.4.29)$$

Proof. Using

$$\mathbb{E}_{\nu_{N,\alpha}} \left[\mathrm{e}^{-\beta|\Lambda_N| H^{(\mathrm{FCH})}_{\mu}} \right] = \mathbb{E}_{\nu^{(\mathrm{PM})}_{N,\mu,\alpha}} \left[\mathrm{e}^{-\beta|\Lambda_N| H_-} \right] Z^{(\mathrm{PM})}_N(\beta,\mu,\alpha), \quad (4.4.30)$$

in conjunction with the LDP in Theorem 4.3.11 and in particular the lower bound (4.3.7), we arrive at

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \left(\mathbb{E}_{\nu^{(\mathrm{PM})}_{N,\mu,\alpha}} \left[\mathrm{e}^{-\beta|\Lambda_N| H_-} \right] Z^{(\mathrm{PM})}(\beta,\mu,\alpha) \right) \\ &\qquad \ge - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I^{(\mathrm{PM})}_{\mu,\alpha}(x) + \beta H_-(x) \right\} - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I^{(\mathrm{PM})}_{\mu,\alpha}(x) \right\} && (4.4.31) \\ &\qquad = - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I(x) + \beta H^{(\mathrm{PM})}_{\mu+\alpha,a,\mathrm{l.s.c.}}(x) + \beta H_-(x) \right\} && (4.4.32) \\ &\qquad = - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I(x) + \beta H^{(\mathrm{FCH})}_{\mu+\alpha,\mathrm{l.s.c.}}(x) \right\}, && (4.4.33) \end{aligned}$$

where the last equality follows from Lemma 4.4.6 below.

□

Lemma 4.4.6.

$$\inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I(x) + \beta H^{(\mathrm{PM})}_{\mu,a,\mathrm{l.s.c.}}(x) + \beta H_-(x) \right\} = \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I(x) + \beta H^{(\mathrm{FCH})}_{\mu,\mathrm{l.s.c.}}(x) \right\} \quad (4.4.34)$$

Proof. The infimum of any function on an open set is equal to the infimum of its lower semicontinuous regularisation over the same set. Recall that $\ell_1(\mathbb{R}_+)$ is open because it is the whole space. We thus need to show that

$$\left(H^{(\mathrm{PM})}_{\mu,a,\mathrm{l.s.c.}} + H_- \right)_{\mathrm{l.s.c.}}(x) = H^{(\mathrm{FCH})}_{\mu,\mathrm{l.s.c.}}(x) \quad \text{for all } x \in \ell_1(\mathbb{R}_+). \quad (4.4.35)$$

Note that

$$H_{\mu}^{(FCH)}(x) = H_{\mu,a}^{(PM)}(x) + H_{-}(x) \geq H_{\mu,a,\text{l.s.c.}}^{(PM)}(x) + H_{-}(x) \geq \left(H_{\mu,a,\text{l.s.c.}}^{(PM)} + H_{-}\right)_{\text{l.s.c.}}(x). \tag{4.4.36}$$

To show (4.4.35) we use the proof of Lemma 4.4.3 and choose the sequence according to (4.4.26). We thus obtain

$$H_{\mu,\text{l.s.c.}}^{(FCH)}(x) \geq \left(H_{\mu,a,\text{l.s.c.}}^{(PM)} + H_{-}\right)_{\text{l.s.c.}}(x). \tag{4.4.37}$$

□

We finally combine Proposition 4.4.4 and Proposition 4.4.5 to finish the proof for Theorem 4.4.1.

□

Chapter 5

Variational Analysis, Pressure Representations and BEC for Cycle Count Models

In Chapter 4 we derived large deviation principles for the empirical cycle counts on $\ell_1(\mathbb{R}_+)$. The main advantage of this over a large deviation principle for the empirical total particle count is that it gives us information on how the particles are distributed amongst the different cycle types. In this chapter we begin by investigating this distribution by looking at the minimisers of the rate functions associated with our large deviation principles. We also use our large deviation principles to find explicit expressions for the thermodynamic pressure in each of our various models. In [Gir60], an order parameter for Bose-Einstein condensation is suggested. Here we provide a related function, where the role of the momentum eigenstate has been replaced with the cycle length. We discuss this "condensate" in the context of each of our models.

As in Chapter 4, for the sake of clarity in our proofs we will set $\beta = 1$ and suppress it from the notation. Once we have these results, it is usually a matter of dimensional analysis to reintroduce it.

5.1 Minimisers of the LDP Rate Functions

We have shown that the ideal Bose gas model, the cycle mean-field (CM) model, the particle mean-field (PM) model and the full cycle HYL (FCH) model all satisfy a LDP for empirical cycle counts with the rate functions I_α, $I^{(CM)}_\alpha$, $I^{(PM)}_{\mu,\alpha}$, and $I^{(FCH)}_{\mu,\alpha}$ respectively. We summarise our results on the zeroes in the following statements.

Proposition 5.1.1. *The rate function for the ideal Bose gas model, I_α, has a unique zero $\xi \in \ell_1(\mathbb{R}_+)$ given by*

$$\xi_k = q_k \mathrm{e}^{\beta\alpha k}, \quad k \in \mathbb{N}. \tag{5.1.1}$$

PROOF. To find the zeroes of the ideal gas rate function, first let us find the critical points by setting the Gâteaux derivative of the function to zero. That is, we find the set of points $\tilde{x} \in \ell_1(\mathbb{R}_+)$ such that

$$\mathrm{d}I_\alpha(\tilde{x}; y) = 0 \quad \forall y \in \ell_1(\mathbb{R}). \tag{5.1.2}$$

This yields a single equation for each element of the sequence $\tilde{x}$. This set of equations has the unique solution $\tilde{x} = \xi$ given in the proposition. Since the rate function I_α is strictly convex where it is finite, this critical point is the unique global minimiser.

□

Proposition 5.1.2. *The rate function $I^{(CM)}_\alpha$ has a unique zero at $\xi^{(CM)} \in \ell_1(\mathbb{R}_+)$ given by*

$$\xi^{(CM)}_k = \frac{W_0(K)}{K} q_k \mathrm{e}^{\beta\alpha k}, \quad k \in \mathbb{N}, \tag{5.1.3}$$

where W_0 is the real branch of the Lambert W function for non-negative arguments, and $K = K(a, \beta, \mu)$ is a dimensionless quantity defined by

$$K := a\beta\overline{q}^{(\alpha)} = \frac{a\beta}{(4\pi\beta)^{\frac{d}{2}}} g\left(1 + \frac{d}{2}, -\beta\alpha\right). \tag{5.1.4}$$

Remark 5.1.3. *Definition and properties of the Bose function, $g(n, x)$, and the Lambert W function are given in Appendices C.1 and C.2.* ◇

PROOF OF PROPOSITION 5.1.2. For the existence of a minimiser, recall that $I^{(CM)}_\alpha$ is lower semicontinuous and has compact level-sets. Also note that I_α is strictly convex where it is finite and $H^{(CM)}$ is convex. Therefore $I_\alpha^{(CM)}$ is strictly convex where it is finite (a non-empty set) and uniqueness of the minimiser follows.

To calculate the minimiser, we search for stationary points. Since $I_\alpha^{(CM)}$ is strictly convex where it is finite, if we find a stationary point then it is the global minimiser. By considering the coordinate derivatives, we know that the minimiser must satisfy all the following equations:

$$\log \frac{x_k}{q_k \mathrm{e}^{\alpha k}} + a \sum_{k=1}^{\infty} x_k = 0, \quad k \in \mathbb{N}. \tag{5.1.5}$$

To make this more manageable, we introduce the dummy variable $\Gamma \in \mathbb{R}_+$ and corresponding equation $\Gamma = \sum_{k=1}^{\infty} x_k$. Our problem is then to solve

$$\log \frac{x_k}{q_k \mathrm{e}^{\alpha k}} + a\Gamma = 0, \quad k \in \mathbb{N}, \tag{5.1.6}$$

$$\Gamma - \sum_{k=1}^{\infty} x_k = 0. \tag{5.1.7}$$

Given Γ, (5.1.6) is uniquely solved by $x_k = q_k \mathrm{e}^{\alpha k} \exp(-a\Gamma), k \in \mathbb{N}$, and therefore (5.1.7) becomes

$$\Gamma \exp(a\Gamma) = \overline{q}^{(\alpha)}. \tag{5.1.8}$$

This has the unique solution $\Gamma = \frac{1}{a} W_0\left(a\overline{q}^{(\alpha)}\right) = \frac{1}{a} W_0(K)$, and so (5.1.6) and (5.1.7) are uniquely jointly solved by $x = \xi$ given by

$$\xi_k = q_k \mathrm{e}^{\alpha k} \exp(-W_0(K)) = \frac{W_0(K)}{K} q_k \mathrm{e}^{\alpha k}, \qquad k \in \mathbb{N}. \tag{5.1.9}$$

□

Proposition 5.1.4. *The rate function $I_{\mu,\alpha}^{(PM)}$ has a unique zero at $\xi^{(PM)} \in \ell_1(\mathbb{R}_+)$ where*

$$\xi_k^{(PM)} = q_k \exp\left(\beta k \left(\mu + \alpha - a\delta^*\right)_-\right), \quad k \in \mathbb{N}, \tag{5.1.10}$$

and $\delta^* = \delta^*(\beta, \mu + \alpha, a)$ is given implicitly as the unique solution to the equation

$$\delta^* = \sum_{k=1}^{\infty} k q_k \exp\left(\beta k \left(\mu + \alpha - a\delta^*\right)_-\right). \tag{5.1.11}$$

Remark 5.1.5. *Note that Proposition 5.1.4 tells us that the zero of* $I^{(PM)}_{\mu,\alpha}$ *is equal to the zero of* I_η*, where* $\eta = (\mu + \alpha - a\delta^*)_-$. ◇

Proof of Proposition 5.1.4. Without loss of generality we put $\alpha \equiv 0$, and write $I_0 = I$. To obtain the unique zero of the rate function we shall find the unique minimiser of the un-normalised rate function $F(x) := I(x) + H^{(PM)}_{\mu,\text{l.s.c.}}(x)$. For the existence of a minimiser, recall that F is lower semicontinuous and has compact level-sets. Also note that I is strictly convex where it is finite, and $H^{(PM)}_{\mu,\text{l.s.c.}}$ is also convex in the linear function $D(x)$. Therefore F is strictly convex where it is finite (a non-empty set) and uniqueness of the minimiser follows. To calculate the minimiser, we search for stationary points. Since F is strictly convex where it is finite, if we find a stationary point then it is the global minimiser. By considering again as in the proof of Proposition 5.1.1 the coordinate derivatives, we know that the minimiser must satisfy all the following equations

$$\log \frac{x_k}{q_k} + k\,(aD(x) - \mu)_+ = 0, \quad k \in \mathbb{N}. \tag{5.1.12}$$

To make this more manageable, we introduce the dummy variable $\delta \in \mathbb{R}_+$ and corresponding equation $\delta = D(x)$.

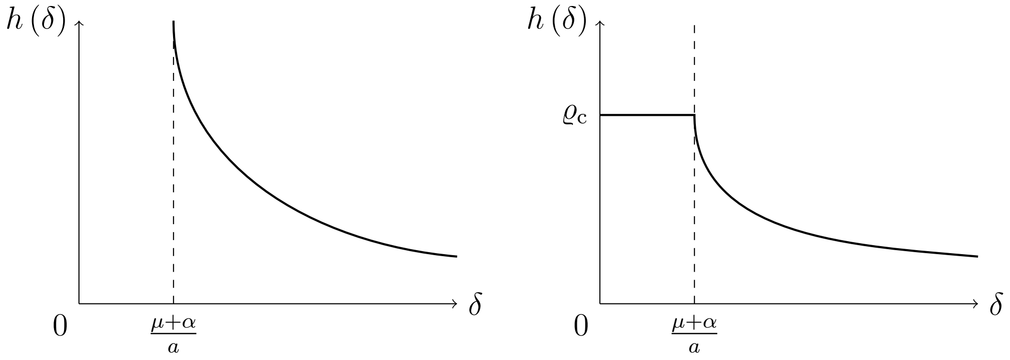
$$\log \frac{x_k}{q_k} + k\,(a\delta - \mu)_+ = 0, \quad k \in \mathbb{N}, \tag{5.1.13}$$

$$\delta - D(x) = 0. \tag{5.1.14}$$

Given the value δ, (5.1.13) is uniquely solved by $x_k = q_k \exp\big(k\,(\mu - a\delta)_-\big)\,, k \in \mathbb{N}$, and therefore (5.1.14) becomes

$$\delta = \sum_{k=1}^{\infty} k q_k \exp\big(\beta k\,(\mu - a\delta)_-\big)\,. \tag{5.1.15}$$

Denote the right hand side in (5.1.15) by $h(\delta)$, and note that $h(\delta) \to 0$ as $\delta \to \infty$.



(a) $d = 1, 2$ (b) $d \geq 3$

Figure 5.1: Sketch of $h(\delta)$. This shows $\mu + \alpha > 0$, but the sketch translates with μ.

Furthermore,

$$\lim_{\delta \to 0} h(\delta) = \begin{cases} \sum_{k \in \mathbb{N}} k q_k \mathrm{e}^{\mu k} \in (0, \infty) & : \mu < 0, \\ \infty & : d = 1, 2 \text{ and } \mu \geq 0, \\ \varrho_{\mathrm{c}} \in (0, \infty) & : d \geq 3 \text{ and } \mu \geq 0, \end{cases} \tag{5.1.16}$$

see Figure 5.1. In all cases h is non-increasing and so there exists a unique solution, which we denote δ^*.

□

The zeroes for the FCH model are more complex and involved.

Proposition 5.1.6. *The zeroes $\xi \subset \ell_1(\mathbb{R}_+)$ of rate the function $I^{(\text{FCH})}_{\mu,\alpha}$ satisfy the following expression,*

$$\xi_k = -\frac{1}{b\beta k^2} W_{\chi_k^*} \left(-b\beta k^2 q_k \exp\left[\beta k \left(\mu + \alpha - a\delta^* \right) \begin{Bmatrix} 1 & : a\delta^* \geq \mu + \alpha \\ -\frac{b}{a-b} & : a\delta^* \leq \mu + \alpha \end{Bmatrix} \right] \right), \tag{5.1.17}$$

for $k \in \mathbb{N}$, where $(\delta^*, \chi^*) \in \mathbb{R}_+ \times \{0, -1\}^{\mathbb{N}}$ is a solution to $\delta = g^{\chi}(\delta)$, where

$$g^{\chi}(\delta) := -\frac{1}{b\beta} \sum_{j=1}^{\infty} \frac{1}{j} W_{\chi_j} \left(-b\beta j^2 q_j \exp\left[\beta j \left(\mu + \alpha - a\delta \right) \begin{Bmatrix} 1 & : a\delta \geq \mu + \alpha \\ -\frac{b}{a-b} & : a\delta \leq \mu + \alpha \end{Bmatrix} \right] \right). \tag{5.1.18}$$

PROOF. Without loss of generality we set $\alpha = 0$ and write $I^{(\text{FCH})}_{\mu,0} = I^{(\text{FCH})}_{\mu}$. For the existence of such a minimiser ξ, recall that $I^{(FCH)}_{\mu,\alpha}$ is lower semicontinuous and has compact level-sets.

Suppose that for a given k, the partial derivative $\frac{\partial I^{(FCH)}_{\mu}}{\partial x_k}$ is defined and non-zero at $x \in \operatorname{int}\{\ell_1(\mathbb{R}_+)\}$. Then $I^{(FCH)}_{\mu}$ does not achieve its infimum at x. Since the boundary

$$\partial \ell_1(\mathbb{R}_+) = \{x \in \ell_1(\mathbb{R}_+) : \exists\, k \text{ s.t. } x_k = 0\}, \tag{5.1.19}$$

and $\frac{\partial I^{(FCH)}_{\mu}}{\partial x_k} = -\infty$ here, the infimum is not achieved here.

For $x \in \operatorname{int}\{\ell_1(\mathbb{R}_+)\}$ we have

$$\frac{\partial I^{(FCH)}_{\mu}}{\partial x_k}(x) = \log \frac{x_k}{q_k} - bk^2 x_k - k(\mu - aD(x)) \left\{ \begin{array}{ll} 1 & : aD(x) \geq \mu \\ -\frac{b}{a-b} & : aD(x) \leq \mu \end{array} \right\}, \qquad k \in \mathbb{N}, \tag{5.1.20}$$

which is defined everywhere in $\operatorname{int}\{\ell_1(\mathbb{R}_+)\}$. Hence a solution ξ must solve $\frac{\partial I^{(FCH)}_{\mu}}{\partial x_k}(\xi) = 0$ for all $k \in \mathbb{N}$. To make this more manageable, we introduce the dummy variable $\delta \in \mathbb{R}_+$ and corresponding equation $\delta = D(x)$. Our problem is then to solve

$$\log \frac{x_k}{q_k} - bk^2 x_k - k(\mu - a\delta) \left\{ \begin{array}{ll} 1 & : a\delta \geq \mu \\ -\frac{b}{a-b} & : a\delta \leq \mu \end{array} \right\} = 0, \qquad k \in \mathbb{N}, \tag{5.1.21}$$

$$\delta - D(x) = 0. \tag{5.1.22}$$

Unfortunately - unlike in the corresponding PM case - even when we are given δ we are not guaranteed to have a solution for (5.1.21), or that such a solution would be unique. If we fix δ, then the k^{th} equation of (5.1.21) either has no solution or is solved by

$$x_k = -\frac{1}{b\beta k^2} W_{\chi_k} \left(-bk^2 q_k \exp\left[k(\mu - a\delta) \left\{ \begin{array}{ll} 1 & : a\delta \geq \mu \\ -\frac{b}{a-b} & : a\delta \leq \mu \end{array} \right\} \right] \right) \tag{5.1.23}$$

for all $\chi_k \in \{0, -1\}$, where W_0 and W_{-1} are the two real branches of the Lambert W function. The 'no solution' case corresponds precisely to W_0 and W_{-1} not being defined for this input.

Substituting these x_k back into (5.1.22) gives the condition (5.1.18) as re-

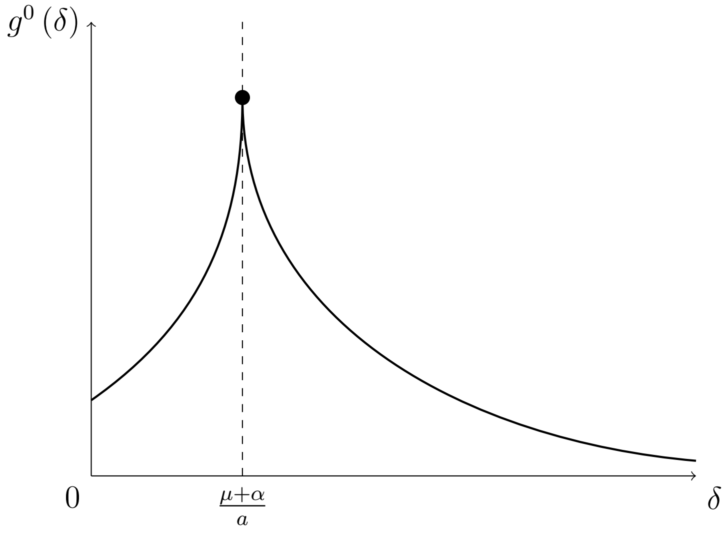


Figure 5.2: Sketch of $g^{\chi}(\delta)$ for $\chi = 0$, $d = 3, 4$, and $\beta \le \left({b^2 e^2}/{(4\pi)^d}\right)^{\frac{1}{d-2}}$. This shows $\mu + \alpha > 0$, but the sketch translates with μ.

quired.

$\square$

The next proposition shows that there exists a regime for the parameters β, α, μ, a, b, and d such that the rate function has a unique zero.

Proposition 5.1.7. *There exists $\tilde{\mu} = \tilde{\mu}(d, \beta, a, b) \in \mathbb{R}$ such that for $\mu + \alpha < \tilde{\mu}$ the rate function $I^{(\text{FCH})}_{\mu,\alpha}$ has a unique zero at $\xi^{(\text{FCH})} \in \ell_1(\mathbb{R}_+)$ where*

$$\xi^{(\text{FCH})}_k = -\frac{1}{b\beta k^2} W_0\left(-b\beta k^2 q_k \exp\left[\beta k\left(\mu + \alpha - a\delta^*\right)\right]\right), \qquad k \in \mathbb{N}, \tag{5.1.24}$$

and $\delta^* = \delta^*(\beta, \mu + \alpha, a, b)$ is given implicitly as the unique solution to the equation $\delta^* = g^0(\delta^*)$, where $g^0 = g^{\chi \equiv 0}$.

Proof. The essence of this proof is to find a set of parameters for which Proposition 5.1.6 produces only one candidate zero. First note that where they are finite, g^{χ} are continuous. Also, for $\chi \neq 0$ we have $g^{\chi}(\delta) \sim C_{\chi}\delta$ as $\delta \to +\infty$ where $C_{\chi} \ge {a}/{b} > 1$. Because g^{χ} has a translational symmetry with $\mu + \alpha$, this means that for sufficiently negative $\mu + \alpha$ these branches will have no solution. In contrast, g^0 is continuous and decreasing for $\delta > {\mu + \alpha}/{a}$. This means that as we decrease $\mu + \alpha$ we are eventually guaranteed to have a unique solution for δ.

$\square$

5.2 Thermodynamic Pressure Representation

We derive representations for the pressure in each of our models. Using our large deviation principles in Section 2.3 and the zeroes of the rate functions in conjunction with our representation of the partition functions in Proposition 2.2.9 we obtain the thermodynamic limit of the pressure in our various models.

Theorem 5.2.1 (Pressure representations). *Let* $\beta > 0$, $\alpha \leq 0$ *and* $\mathrm{bc} \in \{\varnothing, \mathrm{Dir}\}$, *then*

$$p(\beta,\alpha) = \lim_{N\to\infty} \frac{1}{\beta|\Lambda_N|} \log \mathrm{e}^{|\Lambda_N|\overline{q}^{(\mathrm{bc},\alpha)}} = \frac{1}{\beta(4\pi\beta)^{\frac{d}{2}}} \sum_{k=1}^{\infty} \frac{\mathrm{e}^{\beta\alpha k}}{k^{1+d/2}}, \tag{5.2.1}$$

$$\begin{aligned} p^{(\mathrm{CM})}(\beta,\alpha) &= p(\beta,\alpha) + \lim_{N\to\infty} \frac{1}{\beta|\Lambda_N|} \log Z_N^{(\mathrm{CM})}(\beta,\alpha) \\ &= \frac{1}{a\beta^2} W_0\left(a\beta\overline{q}^{(\alpha)}\right)\left(1 + \frac{1}{2} W_0\left(a\beta\overline{q}^{(\alpha)}\right)\right), \\ &\quad \text{with } a \geq 0, \end{aligned} \tag{5.2.2}$$

$$\begin{aligned} p^{(\mathrm{PM})}(\beta,\mu,\alpha) &= p(\beta,\alpha) + \lim_{N\to\infty} \frac{1}{\beta|\Lambda_N|} \log Z_N^{(\mathrm{PM})}(\beta,\mu,\alpha) \\ &= \begin{cases} \frac{a}{2}\delta^{*2} + \frac{1}{\beta}\sum_{k=1}^{\infty} q_k \exp\left(\beta k\left(\mu+\alpha - a\delta^*\right)\right) & : \mu+\alpha \leq a\varrho_{\mathrm{c}}, \\ \frac{(\mu+\alpha)^2}{2a} + \frac{1}{\beta}\sum_{k=1}^{\infty} q_k & : \mu+\alpha \geq a\varrho_{\mathrm{c}}, \end{cases} \\ &\quad \text{with } \mu \in \mathbb{R} \text{ and } a \geq 0, \end{aligned} \tag{5.2.3}$$

$$\begin{aligned} p^{(\mathrm{FCH})}(\beta,\mu,\alpha) &= p(\beta,\alpha) + \lim_{N\to\infty} \frac{1}{\beta|\Lambda_N|} \log Z_N^{(\mathrm{FCH})}(\beta,\mu,\alpha) \\ &= p(\beta,0) - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I_0(x) + \beta H^{(\mathrm{FCH})}_{\mu+\alpha,\mathrm{l.s.c.}}(x) \right\}, \\ &\quad \text{with } \mu \in \mathbb{R}, \text{ and } a > b \geq 0. \end{aligned} \tag{5.2.4}$$

These hold for only $\alpha < 0$ *if* $\mathrm{bc} = \mathrm{per}$.

Proof. The thermodynamic limit of the average finite-volume pressure exists in all models, see [BR97]. The independence of the thermodynamic limit on the choice of boundary conditions follows with [ACK11], or using [Rob71; AN73], see

Appendix A. The pressure for the ideal Bose gas in (5.2.1) follows easily using (2.2.20) and the independence of the boundary conditions.

For (5.2.2), (5.2.3), and (5.2.4) our large deviation principles give us

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z_N^{(\text{CM})}(\beta,\alpha) = p(\beta,\alpha) - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I_\alpha(x) + \beta H^{(\text{CM})}(x) \right\}, \tag{5.2.5}$$

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z_N^{(\text{PM})}(\beta,\mu,\alpha) = p(\beta,0) - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I_0(x) + \beta H^{(\text{PM})}_{\mu+\alpha,\text{l.s.c.}}(x) \right\}, \tag{5.2.6}$$

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z_N^{(\text{FCH})}(\beta,\mu,\alpha) = p(\beta,0) - \inf_{x\in\ell_1(\mathbb{R}_+)} \left\{ I_0(x) + \beta H^{(\text{FCH})}_{\mu+\alpha,\text{l.s.c.}}(x) \right\}. \tag{5.2.7}$$

Here we have used $I_\alpha(x) = I_0(x) - \beta\alpha D(x) + p(\beta,\alpha) - p(\beta,0)$. Then for (5.2.2) and (5.2.3) we substitute in the zeroes found in Proposition 5.1.2 and Proposition 5.1.4. The ideal Bose gas pressure $p(\beta,\alpha)$ appears in all pressure formulae due to the term in (2.2.20) stemming from the reference Poisson process.

□

The following Lemma will prove useful in deriving properties of these pressures from the zeroes of the rate functions found in Section 5.1.

Lemma 5.2.2. *Let $I \subset \mathbb{R}$ be an open interval, $F : \ell_1 \times I \to \mathbb{R}$, and $\xi \in C^1(I;\ell_1)$. Also define*

$$\mathcal{G} : I \to \mathbb{R}; s \mapsto F(\xi(s), s).$$

Then if $F(x,s)$ is Gâteaux differentiable in its first argument at $\xi(s)$ with $\left.\frac{\partial F}{\partial x_k}\right|_\xi = 0$ $\forall k \in \mathbb{N}$, then

$$\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}s} = \left.\frac{\partial F}{\partial s}\right|_\xi. \tag{5.2.8}$$

Proof. An application of the chain rule gives

$$\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}s} = \left.\frac{\partial F}{\partial s}\right|_\xi + \sum_{j=1}^{\infty} \frac{\mathrm{d}\xi_k}{\mathrm{d}s} \left.\frac{\partial F}{\partial x_k}\right|_{x=\xi}. \tag{5.2.9}$$

Since the partial derivatives of F with respect to x_k vanish at ξ, we only keep the first term.

□

In the following sections we study our different models in terms of their thermodynamic behaviour and the onset of Bose-Einstein condensation. One crucial observation is that the partial derivative of the pressure with respect to the chemical potential μ at inverse temperature β equals the expected physical particle density at equilibrium in the grand canonical ensemble at inverse temperature β and chemical potential μ. In the following, we will distinguish between different regimes depending on whether the expected particle density equals the density of the zeroes of the rate functions or not. In the latter case, the excess density equals the density of the condensate in the BEC state.

Thermodynamics of the ideal Bose gas

We collect well-known properties of the ideal gas pressure for convenience of the reader and comparison purposes to our other models, for more details see [BLP88] and [BCMP05].

Proposition 5.2.3. *1. For $\beta > 0, \alpha > 0$, we define $p(\beta, \alpha) = +\infty$. Then the pressure function $p(\beta, \cdot)$ is a convex function on $\mathbb{R}$ with closed level sets: $\{\alpha \in \mathbb{R} : p(\beta, \alpha) \leq \gamma\}$.*

2. For $\beta > 0, \alpha < 0$, the ideal gas pressure $p(\beta, \alpha)$ is smooth with respect to α. In particular,

$$\frac{\mathrm{d}p}{\mathrm{d}\alpha} = D\left(\left\{q_k \mathrm{e}^{\beta\alpha k}\right\}\right). \tag{5.2.10}$$

Proof. (1) Clearly, $p(\beta, 0) = \frac{1}{\beta(4\pi\beta)^{\frac{d}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{1+d/2}} < \infty$ for all $d \geq 1$. Convexity follows from properties of the Bose functions, $g(1 + {}^{d}/_{2}, -\beta\alpha)$, see (C.1.1) in Appendix C.1.

(2) This follows from the Bose functions, $g(n, x)$, being differentiable for $x > 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}x} g(n, x) = -g(n-1, x), \qquad \forall x > 0. \tag{5.2.11}$$

Then the first derivative follows from directly differentiating the representation (5.2.1).

$\square$

We denote by

$$p_{\Lambda_N}^{(\mathrm{bc})}(\beta, \alpha) = \frac{1}{\beta|\Lambda_N|} Z_{\Lambda_N}^{(\mathrm{bc})}(\beta, \alpha) \tag{5.2.12}$$

the average finite-volume pressure with $\mathrm{bc} \in \{\varnothing, \mathrm{per}, \mathrm{Dir}\}$.

Proposition 5.2.4. *1. For $\beta > 0, \alpha < 0$, and any $N \in \mathbb{N}$,*

$$\frac{1}{|\Lambda_N|}\mathsf{E}_\alpha^{(\mathrm{bc})}\left[N_{\Lambda_N}^{(\ell)}\right] = \sum_{k\in\mathbb{N}} k q_{N,k}^{(\mathrm{bc})}\mathrm{e}^{\beta\alpha k} = \frac{\mathrm{d}}{\mathrm{d}\alpha} p_{\Lambda_N}^{(\mathrm{bc})}(\beta,\alpha). \tag{5.2.13}$$

The function $\alpha \mapsto \frac{\mathrm{d}}{\mathrm{d}\alpha} p_{\Lambda_N}^{(\mathrm{bc})}(\beta,\alpha)$ is increasing on $(-\infty, 0)$. It follows that we can give $\frac{1}{|\Lambda_N|}\mathsf{E}_\alpha^{(\mathrm{bc})}[N_{\Lambda_N}^{(\ell)}]$ any pre-assigned value $\varrho \in (0,\infty)$ by choosing $\alpha = \mu_N(\varrho) \in (-\infty, 0)$.

2. In the thermodynamic limit $N \to \infty$,

$$\varrho_{\mathrm{c}} := \lim_{\alpha\uparrow 0}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha} p_{\Lambda_N}^{(\mathrm{bc})}(\beta,\alpha)\right) = \begin{cases} +\infty & : d = 1, 2, \\ \frac{1}{(4\pi\beta)^{\frac{d}{2}}}\zeta\left(\frac{d}{2}\right) & : d \geq 3. \end{cases} \tag{5.2.14}$$

Let $\mu_N(\varrho)$ denote the unique root of

$$\left(\frac{\mathrm{d}}{\mathrm{d}\alpha} p_{\Lambda_N}^{(\mathrm{bc})}(\beta,\alpha)\right) = \varrho \tag{5.2.15}$$

then $\mu(\varrho) = \lim_{N\to\infty}\mu_N(\varrho)$ exists and is equal to the unique root of

$$\left(\frac{\mathrm{d}}{\mathrm{d}\alpha} p^{(\mathrm{bc})}(\beta,\alpha)\right) = \varrho \quad \textit{if } \varrho < \varrho_{\mathrm{c}}, \tag{5.2.16}$$

and it is equal to zero otherwise.

Proof. (1) This follows by direct computation. The exponential term ensures that the derivative of the finite-volume pressure is increasing in μ. As long as the box Λ_N has finite volume one can give the average particle density any pre-assigned value by choosing a chemical potential.
(2) The limit in (5.2.14) is obtained by direct calculation in conjunction with basic properties of the Bose function summarised in Appendix C.1. The convergence of the unique root is ensured as long as the expected particle density stays below the critical density which is finite only in dimensions $d \geq 3$.

□

Proposition 5.2.5. *For $\varrho > 0$, we define the ideal Bose gas free energy as the Legendre-Fenchel transform of the pressure,*

$$f(\beta,\varrho) := \sup_{s\in\mathbb{R}} \{s\varrho - p(\beta,s)\} = \begin{cases} \frac{-1}{\beta(4\pi\beta)^{\frac{d}{2}}} g(1+\frac{d}{2}, -\beta\alpha) + \varrho\alpha & : \varrho \le \varrho_{\mathrm{c}} \,, \\ \frac{-1}{\beta(4\pi\beta)^{\frac{d}{2}}} \zeta\left(1+\frac{d}{2}\right) & : \varrho \ge \varrho_{\mathrm{c}} \,, \end{cases} \tag{5.2.17}$$

where $\alpha \le 0$ is a solution to

$$\frac{1}{(4\pi\beta)^{\frac{d}{2}}} g\left(\frac{d}{2}, -\beta\alpha\right) = \varrho, \tag{5.2.18}$$

which exists and is unique for $\varrho \le \varrho_{\mathrm{c}}$.

Proof. Since $p(\beta,s) = +\infty$ for $s > 0$, we only need to search $s \le 0$. On the interior of this region p is differentiable, and we look for stationary points. If $\varrho \ge \varrho_{\mathrm{c}}$, then there are no stationary points for $s < 0$ and $s\varrho - p(\beta,s)$ is increasing in s. Hence the supremum is achieved at $s = 0$. If $\varrho < \varrho_{\mathrm{c}}$, then there is a unique stationary point. This is also a local maximum and is given at $s = \alpha$ as required. This has the required limit as $\varrho \uparrow \varrho_{\mathrm{c}}$ implying the continuity for f.

□

Remark 5.2.6. *It is easy to see that $\varrho \mapsto f(\beta,\varrho)$ is a decreasing convex function. It is given by*

$$f(\beta,\varrho) = \mu(\varrho)\varrho - p(\beta,\mu(\varrho)) \quad \textit{for } \varrho < \varrho_{\mathrm{c}}. \tag{5.2.19}$$

The linear segment in the graph of f where f is constant and equal to $-p(\beta,0)$ for $\varrho \ge \varrho_{\mathrm{c}}$, signals a first-order phase-transition at chemical potential $\alpha = 0$. For a derivation of this free energy directly fom the canonical ensemble, see [Ada08]. ◇

Thermodynamics of the CM model

We summarise our findings for the CM model below.

Proposition 5.2.7. *1. For $\beta > 0$, we define $p^{(CM)}(\beta,\alpha) = +\infty$ for $\alpha > 0$. Then the pressure $p^{(CM)}(\beta,\cdot)$ is a is a convex function on $\mathbb{R}$ with closed level sets: $\{\alpha \in \mathbb{R} : p^{(CM)}(\beta,\alpha) \le \gamma\}$.*

2. For $\beta > 0$, $\alpha < 0$, the CM gas pressure $p^{(CM)}(\beta,\alpha)$ is smooth with respect to α. In particular,

$$\frac{\mathrm{d}p^{(CM)}}{\mathrm{d}\alpha} = D\left(\xi^{(CM)}\right) = \frac{W_0\left(a\beta\overline{q}^{(\alpha)}\right)}{a\beta\overline{q}^{(\alpha)}} D\left(\left\{q_k \mathrm{e}^{\beta\alpha k}\right\}\right). \tag{5.2.20}$$

3. In the thermodynamic limit $N \to \infty$,

$$\varrho_{\mathrm{c}}^{(CM)} := \lim_{\alpha\uparrow 0}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha} p_{\Lambda_N}^{(CM)}(\beta,\alpha)\right) = \begin{cases} +\infty & : d = 1, 2, \\ \frac{W_0\left(a\beta\overline{q}^{(0)}\right)}{a\beta\overline{q}^{(0)}(4\pi\beta)^{\frac{d}{2}}}\zeta\left(\frac{d}{2}\right) & : d \geq 3. \end{cases} \tag{5.2.21}$$

Proof. (1) The continuity of $p^{(CM)}$ for $\alpha \leq 0$ follows from (5.2.2) and the continuity of $K = a\beta\overline{q}^{(\alpha)}$ and W_0. Convexity follows from considering the derivatives of $W_0(K)$ with respect to μ for $\alpha < 0$. Appendix C.2 is useful for this calculation.

(2) Smoothness follows from W_0 and the Bose functions being differentiable on the appropriate regions. The form of the first derivative can be found by either directly differentiating (5.2.2), or by using Lemma 5.2.2 with the zero found in Proposition 5.1.2.

(3) We obtain (5.2.21) from (5.2.20) and the continuity of W_0 and $\overline{q}^{(\alpha)}$.

□

Proposition 5.2.8. *For $\varrho > 0$, the CM Bose gas free energy is defined as the Legendre-Fenchel transform of the pressure,*

$$f^{(CM)}(\beta,\varrho) := \sup_{s\in\mathbb{R}}\left\{s\varrho - p^{(CM)}(\beta,s)\right\} = \begin{cases} \varrho\mu - p^{(CM)}(\beta,\mu) & : \varrho \leq \varrho_{\mathrm{c}}^{(CM)}, \\ -p^{(CM)}(\beta,0) & : \varrho \geq \varrho_{\mathrm{c}}^{(CM)}, \end{cases} \tag{5.2.22}$$

where μ is a solution to

$$\frac{1}{a\beta\overline{q}^{(\alpha)}} W_0\left(a\beta\overline{q}^{(\alpha)}\right)\sum_{k=1}^{\infty} k q_k \mathrm{e}^{\beta.\alpha k} = \varrho, \tag{5.2.23}$$

which exists and is unique for $\varrho \leq \varrho_{\mathrm{c}}^{(CM)} := \frac{W_0\left(a\beta\overline{q}^{(\alpha)}\right)}{a\beta\overline{q}^{(\alpha)}}\varrho_{\mathrm{c}}$.

Proof. This is proven in the same way as Proposition 5.2.5.

□

Remark 5.2.9. *The CM model shows the same phase transition as the ideal Bose gas in that the free energy is constant in the density beyond its specific critical density. The critical density of the CM model, however, is different from the ideal Bose gas one. Using properties of the Lambert function, see Appendix C.2, we know that*

$$\lim_{c\downarrow 0}\frac{W_0(cx)}{cx}=1, \tag{5.2.24}$$

$$\lim_{c\to\infty}\frac{W_0(cx)}{cx}=0. \tag{5.2.25}$$

Hence, as the coupling parameter $a\to 0$ vanishes, we obtain the critical ideal Bose gas density, and as $a\to\infty$ the critical density decreases indicating BEC for much lower particle densities. When the coupling parameter a increases the number of finite cycles is suppressed in the probability measure, and therefore the system undergoes a transition to a regime where the particle density is realised in so-called infinite cycles. The CM model has not been studied in the literature so far, it shows similar behaviour as the ideal Bose gas because the Hamiltonian adds only weight on large numbers of cycles present. The following models involving the physical particle density have a more complex phase transition behaviour. As in the ideal Bose gas, the condensate density in Theorem 5.3.2 can be computed only in the sub-critical regime. This is again due to the degenerate behaviour of the distribution. ◇

Thermodynamics of the PM Bose Gas

We collect our findings for the PM model. Our results are using rigorous large deviation analysis and all possible ranges of the chemical potential. In addition, we compute the condensate density in Theorem 5.3.3 below. The density square term in the Hamiltonian stabilises the distribution such that the limiting distribution is no longer degenerate, allowing us the compute the partial derivatives for all values of the chemical potential. We identify regimes where the expected particle density equals the density of the rate function zero.

Proposition 5.2.10. *For $\beta > 0$ and $\alpha \leq 0$ (or only $\alpha < 0$ if $\mathrm{bc} = \mathrm{per}$), the pressure $p^{(PM)}(\beta, \cdot, \alpha) \in C^1(\mathbb{R})$ and is convex. In particular,*

$$\mu + \alpha < a\varrho_c \implies \frac{\mathrm{d}p^{(PM)}}{\mathrm{d}\mu} = D\left(\xi^{(PM)}\right) > \frac{\mu + \alpha}{a} \tag{5.2.26}$$

$$\mu + \alpha = a\varrho_c \implies \frac{\mathrm{d}p^{(PM)}}{\mathrm{d}\mu} = D\left(\xi^{(PM)}\right) = \frac{\mu + \alpha}{a} \tag{5.2.27}$$

$$\mu + \alpha > a\varrho_c \implies \frac{\mathrm{d}p^{(PM)}}{\mathrm{d}\mu} = \frac{\mu + \alpha}{a} > D\left(\xi^{(PM)}\right). \tag{5.2.28}$$

Proof. This follows from either directly differentiating (5.2.3), or by using Lemma 5.2.2 with the zero found in Proposition 5.1.4. Convexity also follows from Proposition 5.1.4, noting that $\mu \mapsto D\left(\xi^{(PM)}(\mu)\right)$ is continuous and increasing.

□

Proposition 5.2.11. *For $x > 0$, the PM Bose gas free energy is defined as the Legendre-Fenchel transform of the pressure,*

$$f^{(PM)}(\beta, x) := \sup_{\alpha \leq 0, \mu \in \mathbb{R}} \left\{(\mu + \alpha)x - p^{(PM)}(\beta, \mu, \alpha)\right\} = f(\beta, x) + \frac{a}{2}x^2. \tag{5.2.29}$$

Proof. The proof is obtained directly from our analysis above.

□

Thermodynamics of the FCH model

We collect our findings for the FCH model. Note that our derivation of the LDP and the thermodynamic pressure results hold for all parameter $a > b$ where the derivation of the thermodynamic pressure in [BDLP90b] is applied only to $a = 2b$. We first identify the sub-critical regime where the expected physical particle density equals the density of the possible zeroes of the FCH rate function.

Proposition 5.2.12. *There exists $\tilde{\mu} = \tilde{\mu}(d, \beta, a, b) \in \mathbb{R}$ such that for $\mu + \alpha < \tilde{\mu}$, $p^{(FCH)}(\beta, \mu, \alpha)$ is smooth and convex in μ. In particular,*

$$\frac{\mathrm{d}p^{(FCH)}}{\mathrm{d}\mu} = D\left(\xi^{(FCH)}\right) \tag{5.2.30}$$

for this range of μ. $\xi^{(\text{FCH})}$ is given in Proposition 5.1.7.

Proof. This is proven by using Lemma 5.2.2 with the zero found in Proposition 5.1.7. Convexity also follows from Proposition 5.1.7, noting that $D\left(\xi^{(\text{FCH})}\right)$ is continuous and increasing in μ.

$\square$

The following proposition shows that, for dimension $d = 3, 4$ and large β (depending on the value for the counter energy term with pre-factor b), the pressure is no longer smooth and thus the density pressure relation is void, signalling some critical behaviour.

Proposition 5.2.13. *For $d = 3, 4$, $a > b > 0$ and $\beta \geq \beta^* = \left({b^2 \mathrm{e}^2}/{(4\pi)^d}\right)^{\frac{1}{d-2}}$, the pressure $p^{(\text{FCH})}(\beta, \cdot, \alpha) \notin C^1(\mathbb{R})$.*

Proof. Let us set $\alpha \equiv 0$. Recall that $g^\chi(\delta)$ denotes the right hand side of (5.1.18) for a given χ. Now $\beta \geq \beta^*$ ensures that g^χ can be defined for all $\delta \in \mathbb{R}$. For this proof we will only require $\mu \leq \overline{\mu} := a g^0\left({\mu}/{a}\right) = \frac{-1}{b\beta} \sum_{j=1}^\infty \frac{1}{j} W_0\left(-b\beta j^2 q_j^{(0)}\right)$.

Note that for $d \geq 3$, the arguments for the Lambert W functions are strictly increasing in the summation index k, approaching 0. This means that since the difference $W_0(x) - W_{-1}(x) \geq 0$ is strictly increasing in x and equals 0 if and only if $x = -\mathrm{e}^{-1}$, we only need to consider finitely many χ for a given μ (all of which are eventually 0). Now since any non-convexity in g^χ can only arrive via the finitely many $\chi_k = -1$ terms, solutions to $\delta = g^\chi(\delta)$ are locally finite in $\mathbb{R}$. To complement this, note that $\lim_{\delta \to +\infty} g^0(\delta) = 0$ whilst for $\chi \neq 0$ we have $g^\chi(\delta) \gg \delta$. Hence we only need to consider a finite range of δ, and therefore for a given μ there are only finitely many solutions for δ.

Because g^χ is continuous for each χ, we can collect solutions uniquely and maximally into continuous paths $\xi^j(\mu)$ defined on closed (possibly infinite) intervals I^j with non-empty interior. We allow families to overlap at endpoints of these intervals. Because we are only considering $\mu \leq \overline{\mu}$ and there are only finitely many solutions for each μ, we will only have finitely many families being relevant to our discussion.

For each of these families, we will denote

$$D^j(\mu) := D\left(\xi^j(\mu)\right), \qquad P^j(\mu) := p(\beta,0) - \frac{1}{\beta}\left(I + \beta H^{(FCH)}_{\mu,\mathrm{l.s.c.}}\right)\left(\xi^j(\mu)\right), \quad (5.2.31)$$

defined on the interval I^j. From Proposition 5.1.6 and Theorem 5.2.1 we know that

$$p^{(FCH)}(\beta,\mu) = p(\beta,0) - \frac{1}{\beta}\inf_{\ell_1}\left\{I + \beta H^{(FCH)}_{\mu,\mathrm{l.s.c.}}\right\} = \max_j P^j(\mu). \qquad (5.2.32)$$

Therefore for each μ, there exists a J such that $p^{(FCH)}(\beta,\mu) = P^J(\mu)$.

From the continuity of g^χ we know that all D^j are continuous on their I^j. Then Lemma 5.2.2 tells us that each P^j is differentiable on $\mathrm{int}(I^j)$, with derivative

$$\frac{\mathrm{d}P^j}{\mathrm{d}\mu} = D^j + \frac{(\mu - aD^j)_+}{a-b}. \qquad (5.2.33)$$

Continuity of this derivative follows from the continuity of D^j.

Let us now consider the $\chi = 0$ solutions. Since g^0 is convex when restricted to $\delta \le \mu/a$, $\frac{\mathrm{d}g^\chi}{\mathrm{d}\delta} \to +\infty$ as $\delta \uparrow \mu/a$, and g^0 is decreasing for $\delta \ge \mu/a$, there exists $\underline{\mu} < \overline{\mu}$ such that this branch has multiple solutions if and only if $\mu \in \left[\underline{\mu}, \overline{\mu}\right]$. Let us label these ξ^0, ξ^1, and ξ^2 such that $D^0 \ge D^1 \ge D^2$. Note that $I^0 = (-\infty, \overline{\mu}]$, $I^1 = \left[\underline{\mu}, \overline{\mu}\right]$, and $I^2 = \left[\underline{\mu}, +\infty\right)$. For a visualisation of these solutions, see Figure 5.4.

Since $\xi^0(\overline{\mu}) = \xi^1(\overline{\mu})$ and $\xi^1\left(\underline{\mu}\right) = \xi^2\left(\underline{\mu}\right)$, we have $P^0(\overline{\mu}) = P^1(\overline{\mu})$ and $P^1\left(\underline{\mu}\right) = P^2\left(\underline{\mu}\right)$. Because $D^0 \ge \mu/a$ and $D^{1,2} \le \mu/a$, we have

$$\frac{\mathrm{d}P^0}{\mathrm{d}\mu} = D^0, \qquad \frac{\mathrm{d}P^1}{\mathrm{d}\mu} = \frac{b}{a-b}\left(\frac{\mu}{b} - D^1\right) < \frac{b}{a-b}\left(\frac{\mu}{b} - D^2\right) = \frac{\mathrm{d}P^2}{\mathrm{d}\mu}, \qquad (5.2.34)$$

on $\left(\underline{\mu}, \overline{\mu}\right)$. Together these mean that $P^2(\overline{\mu}) > P^1(\overline{\mu}) = P^0(\overline{\mu})$.

Now extending our attention to all ξ^j defined on some part of $(-\infty, \overline{\mu}]$, we define

$$M := \left\{\mu \le \overline{\mu} : \exists j \text{ such that } P^j(\mu) > P^0(\mu)\right\}. \qquad (5.2.35)$$

We have just shown that $M \ne \varnothing$, so $\hat{\mu} := \inf M$ is finite. Since P^2 and P^0 are continuous, $\hat{\mu} < \overline{\mu}$.

If $\hat{\mu} \in M$, then $\max_j P^j(\mu)$ is discontinuous at $\mu = \hat{\mu}$. In this case we are done.

If $\hat{\mu} \notin M$, then since the P^j are each continuous, $\exists J$ and $\epsilon > 0$ such that $P^J(\mu) > P^0(\mu)$ for $\mu \in (\hat{\mu}, \hat{\mu} + \epsilon)$.

Now we have to show that the derivatives of P^0 and P^J necessarily have different limits as we take $\mu \to \hat{\mu}$ from their respective sides. First note that

$$\lim_{\mu \uparrow \hat{\mu}} \frac{\mathrm{d}P^0}{\mathrm{d}\mu} = D^0(\hat{\mu}). \tag{5.2.36}$$

Note that $g^\chi(\delta) \geq g^0(\delta)$ with equality only if $\chi = 0$ or if we have both $\beta = \beta^*$ and $\delta = {}^{\mu}/_{a}$. Therefore $D^J \neq {}^{\mu}/_{a}$ and

$$D^J < \frac{\mu}{a} \implies D^J \in \left[D^2, D^1\right] \tag{5.2.37}$$

$$D^J > \frac{\mu}{a} \implies D^J > D^0. \tag{5.2.38}$$

This last inequality is strict because equality could only occur at $\mu = \overline{\mu}$, but $\hat{\mu} < \overline{\mu}$.

If $D^J > {}^{\mu}/_{a}$, then

$$\lim_{\mu \downarrow \hat{\mu}} \frac{\mathrm{d}P^J}{\mathrm{d}\mu} = D^J(\hat{\mu}) > D^0(\hat{\mu}) = \lim_{\mu \uparrow \hat{\mu}} \frac{\mathrm{d}P^0}{\mathrm{d}\mu}, \tag{5.2.39}$$

and we are done.

From the symmetry of g^χ about $\delta = {}^{\mu}/_{a}$ and from g^0 being decreasing for $\delta \geq {}^{\mu}/_{a}$, we have

$$D^0 - \frac{\mu}{a} < \frac{b}{a-b}\left(\frac{\mu}{a} - D^1\right) \tag{5.2.40}$$

for $\mu \in \left[\underline{\mu}, \overline{\mu}\right)$. See Figure 5.3. This implies that if $D^J < {}^{\mu}/_{a}$,

$$\frac{\mathrm{d}P^J}{\mathrm{d}\mu} = \frac{b}{a-b}\left(\frac{\mu}{b} - D^J\right) \geq \frac{b}{a-b}\left(\frac{\mu}{b} - D^1\right) > D^0 = \frac{\mathrm{d}P^0}{\mathrm{d}\mu}. \tag{5.2.41}$$

Taking the limit to $\hat{\mu}$ gives our result.

□

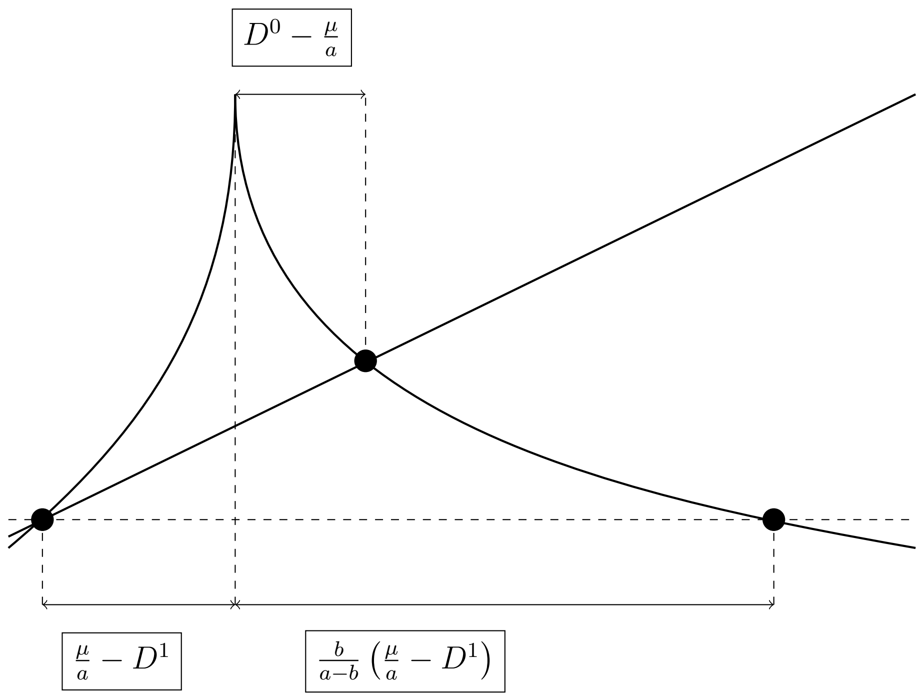


Figure 5.3: Sketch of part of (5.1.18) to show the inequality (5.2.40). Note $\alpha \equiv 0$.

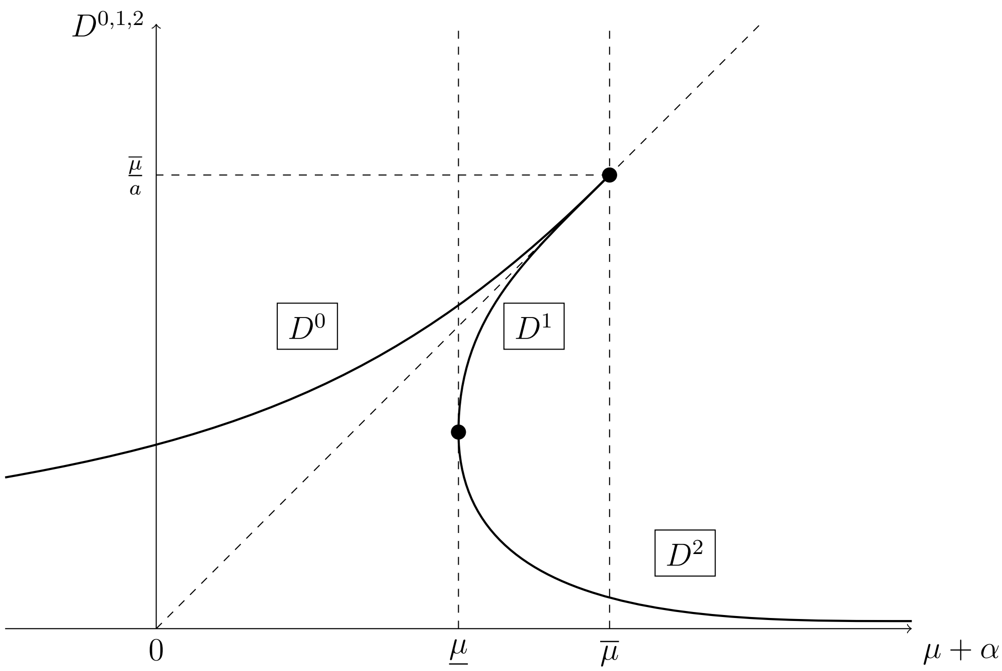


Figure 5.4: Sketch of the total particle density of the three $\chi = 0$ solutions for $d = 3, 4$, $\beta \geq \beta^*$.

5.3 Condensate Analysis

In [Lew86], Lewis considers the "generalised condensation" of [Gir60] as an order parameter for the Bose-Einstein condensation phase transition. Having in mind the density of particles with zero single particle energy in the thermodynamic limit, Lewis first takes the finite volume expected density of particles with energy below some cut-off. He then takes the thermodynamic limit before taking the cut-off to zero. In contrast to Lewis' model, we do not keep track of the particles' energy. As described in Section 2.3.5, we partition our gas by loop type, and expect the condensate to occupy loops of diverging length. Therefore we want to evaluate the 'condensate density' given by

$$\Delta^{(H)}(\beta,\mu) := \lim_{K\to\infty}\lim_{N\to\infty}\mathbb{E}^{(H)}_N\left[D - D_K\right], \qquad D_K(x) := \sum_{j=1}^{K} jx_j, \tag{5.3.1}$$

where the H here indicates whichever model we wish to discuss.

Here we present the condensate density for the ideal Bose gas.

Theorem 5.3.1. *For $\beta > 0$, $\alpha < 0$, we have $\Delta(\beta,\alpha) = 0$.*

For $\beta > 0$, $\alpha = 0$,

$$\Delta(\beta,0) = \begin{cases} +\infty & : d = 1,2, \ \textit{or}\ d \geq 3, \mathrm{bc} = \mathrm{per} \\ 0 & : d \geq 3, \mathrm{bc} \in \{\varnothing, \mathrm{Dir}\}. \end{cases} \tag{5.3.2}$$

Proof. Let us begin with the $\alpha < 0$ case, because a similar approach will be important for our approaches to the PM and FCH cases. For $s \leq -\alpha$, fixed N, and fixed K, define

$$g^{(K)}_N(s) := \frac{1}{\beta|\Lambda_N|}\log\mathbb{E}_{N,\alpha}\left[\exp\left(|\Lambda_N| s\beta(D - D_K)\right)\right]. \tag{5.3.3}$$

Then

$$\left.\frac{\mathrm{d}g_N^{(K)}}{\mathrm{d}s}\right|_{s=0} = \mathbb{E}_{N,\mu}\left[D - D_K\right] \tag{5.3.4}$$

$$\Delta\left(\beta,\alpha\right) = \lim_{K\to\infty}\lim_{N\to\infty}\left.\frac{\mathrm{d}g_N^{(K)}}{\mathrm{d}s}\right|_{s=0}. \tag{5.3.5}$$

Since $g_N^{(K)}$ are all convex in s, we will use Griffith's Lemma to get the point-wise limit of the derivative from the derivative of the point-wise limit:

$$\Delta\left(\beta,\alpha\right) = \lim_{K\to\infty}\frac{\mathrm{d}}{\mathrm{d}s}\left.\left(\lim_{N\to\infty}g_N^{(K)}\left(s\right)\right)\right|_{s=0}. \tag{5.3.6}$$

To calculate the point-wise limit of $g_N^{(K)}$, we first rewrite $g_N^{(K)}$ as

$$g_N^{(K)}\left(s\right) = \frac{1}{\beta\left|\Lambda_N\right|}\log\mathbb{E}_{N,\alpha+s}\left[\exp\left(-\left|\Lambda_N\right| s\beta D_K\right)\right] + \frac{1}{\beta\left|\Lambda_N\right|}\log\frac{Z_N\left(\beta,\alpha+s\right)}{Z_N\left(\beta,\alpha\right)}. \tag{5.3.7}$$

Then we want to use Varadhan's Lemma with our LDP for the ideal Bose gas measure and the tilt $\phi = -s\beta D_K$. This ϕ is continuous, but we need to pay attention to the boundedness conditions. We will show

$$\lim_{M\to\infty}\limsup_{N\to\infty}\frac{1}{\left|\Lambda_N\right|}\mathbb{E}_{N,\alpha+s}\left[\mathrm{e}^{\left|\Lambda_N\right|\phi}\mathbb{1}\left\{\phi \geq M\right\}\right] = -\infty. \tag{5.3.8}$$

For $0 \leq s \leq -\mu$, we have $\phi \leq 0$ almost surely, so (5.3.8) holds trivially. For $s < 0$ we have to work a little harder. Since ϕ is continuous, the set $\{\phi = m\}$ is closed (and measurable). Hence our LDP for the ideal Bose gas model gives us a bound on the probability of this set:

$$\limsup_{N\to\infty}\frac{1}{\left|\Lambda_N\right|}\log\nu_{N,\alpha+s}\left(\phi = m\right) \leq -\inf_{\phi=m} I_{\mu+s} \leq \overline{q}^{(0)} + \beta\left(\alpha+s\right)\left(\frac{m}{\left|s\right|\beta}\right). \tag{5.3.9}$$

This means that for sufficiently large N there exists a m and N independent constant $C > \overline{q}^{(0)}$, such that

$$\mathrm{e}^{m\left|\Lambda_N\right|}\nu_{N,\alpha+s}\left(\phi = m\right) \leq \exp\left(\left|\Lambda_N\right|\left[C + \frac{\mu}{\left|s\right|}m\right]\right). \tag{5.3.10}$$

Since $\alpha < 0$, we have sufficiently fast decay in m to prove that (5.3.8) holds even for $s < 0$, and Varadhan gives us

$$\lim_{N\to\infty} g_N^{(K)}(s) = -\inf_{\ell_1}\left\{\frac{1}{\beta}I_{\alpha+s} + sD_J\right\} + p(\beta,\alpha+s) - p(\beta,\alpha), \qquad \forall s \leq -\alpha. \tag{5.3.11}$$

In the style of Lemma 5.1.1, we can find that this infimum is achieved at $\xi(s) \in \ell_1(\mathbb{R}_+)$, where

$$\xi_k = \begin{cases} q_k \mathrm{e}^{\beta\alpha k} & : k \leq K, \\ q_k \mathrm{e}^{\beta(\alpha+s)k} & : k > K. \end{cases} \tag{5.3.12}$$

Hence

$$\lim_{N\to\infty} g_N^{(K)}(s) = \frac{1}{\beta}\sum_{j=K+1}^{\infty} q_j \mathrm{e}^{\beta\alpha j}\left(\mathrm{e}^{\beta s j} - 1\right), \tag{5.3.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\lim_{N\to\infty} g_N^{(K)}(s)\right)\bigg|_{s=0} = \sum_{j=K+1}^{\infty} j q_j \mathrm{e}^{\beta\alpha j}. \tag{5.3.14}$$

Finally the sum vanishes as $K \to \infty$.

For $\alpha = 0$ with $d = 1, 2$, we take a more direct approach. It is clear from our construction of the ideal Bose gas model that $\lim_{N\to\infty} \mathbb{E}_{N,0}\left[j\boldsymbol{\lambda}_j^{(N)}\right] = jq_j$ for all $j \in \mathbb{N}$. Then for all $M > K$,

$$\liminf_{N\to\infty} \mathbb{E}_{N,0}\left[\sum_{j=K+1}^{\infty} j\boldsymbol{\lambda}_j^{(N)}\right] \geq \lim_{N\to\infty} \mathbb{E}_{N,0}\left[\sum_{j=K+1}^{M} j\boldsymbol{\lambda}_j^{(N)}\right] = \sum_{j=K+1}^{M} jq_j. \tag{5.3.15}$$

Since this lower bound diverges as $M \to \infty$ if $\alpha = 0$ and $d = 1, 2$, we have our result for this case.

For $\alpha = 0$ and $d \geq 3$, the behaviour changes depending upon the boundary condition. Nevertheless, by applying direct methods similar to the $\alpha = 0$ and $d = 1, 2$ case we get the required results.

$\square$

In so far as we can calculate, the condensate density for the CM model looks very similar to the ideal gas.

Theorem 5.3.2. *For all $\beta > 0$, $\alpha < 0$,*

$$\Delta^{(\text{CM})}(\beta, \alpha) = 0. \tag{5.3.16}$$

PROOF. This proof begins identically to Theorem 5.3.1. For $s \le -\mu$, fixed N, and fixed K, define

$$g_N^{(K)}(s) := \frac{1}{\beta |\Lambda_N|} \log \mathbb{E}_{\nu_{N,\mu}^{(\text{CM})}} \left[\exp\left(|\Lambda_N| s\beta \left(D - D_K\right)\right)\right]. \tag{5.3.17}$$

Then by rearranging terms and applying Varadhan, we find that for $s \le -\mu$,

$$\lim_{N\to\infty} g_N^{(K)}(s) = -\inf_{\ell_1} \left\{\frac{1}{\beta} I_{\mu+s}^{(\text{CM})} + sD_J\right\} + p^{(\text{CM})}(\beta, \mu + s) - p^{(\text{CM})}(\beta, \mu). \tag{5.3.18}$$

In the style of Lemma 5.1.2, we can find that this infimum is achieved at $\xi(s) \in \ell_1(\mathbb{R}_+)$, where

$$\xi_k = \begin{cases} \frac{W_0\left(a\beta\overline{q}^{(\mu)}\right)}{a\beta\overline{q}^{(\mu)}} q_k^{(\mu)} & : k \le K, \\ \frac{W_0\left(a\beta\overline{q}^{(\mu+s)}\right)}{a\beta\overline{q}^{(\mu+s)}} q_k^{(\mu+s)} & : k > K. \end{cases} \tag{5.3.19}$$

Substituting this into the infimum, and then taking the derivative gives us

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\lim_{N\to\infty} g_N^{(K)}(s)\right)\bigg|_{s=0} = \frac{W_0\left(a\beta\overline{q}^{(\mu)}\right)}{a\beta\overline{q}^{(\mu)}} \sum_{j=K+1}^{\infty} j q_j^{(\mu)}. \tag{5.3.20}$$

Finally the sum vanishes as $K \to \infty$, and Griffith's Lemma gives us the result.

$\square$

Theorem 5.3.3. *For all $\beta > 0$, $\mu \in \mathbb{R}$ and $\alpha \le 0$,*

$$\begin{aligned} \Delta^{(\text{PM})}(\beta, \mu, \alpha) &= \left(\frac{\mathrm{d}p^{(\text{PM})}}{\mathrm{d}\mu}(\beta, \mu, \alpha) - \varrho_{\mathrm{c}}(\beta)\right)_+ = \left(\frac{\mu + \alpha}{a} - \varrho_{\mathrm{c}}(\beta)\right)_+ \\ &= \left(\frac{\partial}{\partial\mu}\left(H^{(\text{PM})} - H_{\mu+\alpha,\text{l.s.c.}}^{(\text{PM})}\right)\right)\left(\xi^{(\text{PM})}\right), \end{aligned} \tag{5.3.21}$$

where $\xi^{(\text{PM})}$ is the unique minimiser (zero) of the rate function $I_{\mu,\alpha}^{(\text{PM})}$.

PROOF. Let us set $\alpha = 0$ and omit it from the notation. Our proof begins similarly to Theorems 5.3.1 and 5.3.2. For $\mu, s \in \mathbb{R}$, fixed N, and fixed K, define

$$g_N^{(K)}(s) = \frac{1}{\beta |\Lambda_N|} \log \mathbb{E}_{\nu_{N,\mu}^{(PM)}} \left[\exp \left(|\Lambda_N| s\beta (D - D_K) \right) \right]. \tag{5.3.22}$$

We once again rearrange terms to get

$$g_N^{(K)}(s) = \frac{1}{\beta |\Lambda_N|} \log \mathbb{E}_{\nu_{N,\mu+s}^{(PM)}} \left[\exp \left(- |\Lambda_N| s\beta D_K \right) \right] + \frac{1}{\beta |\Lambda_N|} \log \frac{Z_N^{(PM)}(\beta, \mu+s)}{Z_N^{(PM)}(\beta, \mu)}. \tag{5.3.23}$$

Then we want to use Varadhan's Lemma with our LDP for the PM measure and the tilt $\phi = -s\beta D_K$. This ϕ is continuous, but we need to pay attention to the boundedness conditions. We will show

$$\lim_{M \to \infty} \limsup_{N \to \infty} \frac{1}{|\Lambda_N|} \mathbb{E}_{\nu_{N,\mu+s}^{(PM)}} \left[e^{|\Lambda_N| \phi} \mathbb{1} \{ \phi \geq M \} \right] = -\infty. \tag{5.3.24}$$

For $s \geq 0$, we have $\phi \leq 0$ almost surely, so (5.3.24) holds trivially. For $s < 0$ we have to work a little harder. Our LDP for the PM model gives us a bound on the probability of this set:

$$\limsup_{N \to \infty} \frac{1}{|\Lambda_N|} \log \nu_{N,\mu+s}^{(PM)} (\phi = m) \leq - \inf_{\phi = m} I_{\mu+s}^{(PM)} \tag{5.3.25}$$

$$\frac{m}{|s| \beta} \geq \frac{\mu+s}{a} \implies \qquad \leq \inf_{\ell_1} \left\{ I + \beta H_{\mu+s,\text{l.s.c.}}^{(PM)} \right\} + \beta(\mu+s) \left(\frac{m}{|s|\beta} \right) - \frac{a\beta}{2} \left(\frac{m}{|s|\beta} \right)^2. \tag{5.3.26}$$

This means that given $m \geq |s| \beta \frac{\mu+s}{a}$, then for sufficiently large N there exists a m and N independent constant $C > \inf_{\ell_1} \left\{ I_0 + \beta H_{\mu+s,\text{l.s.c.}}^{(PM)} \right\}$, such that

$$e^{m|\Lambda_N|} \nu_{N,\mu+s}^{(PM)} (\phi = m) \leq \exp \left(|\Lambda_N| \left[C + \frac{\mu}{|s| m} - \frac{a}{2\beta |s|^2} m^2 \right] \right). \tag{5.3.27}$$

The very fast decay with m proves that (5.3.24) holds even for $s < 0$, and Varadhan gives us

$$\lim_{N \to \infty} g_N^{(K)}(s) = - \inf_{\ell_1} \left\{ \frac{1}{\beta} I_{\mu+s}^{(PM)} + sD_J \right\} + p^{(PM)}(\beta, \mu+s) - p^{(PM)}(\beta, \mu), \qquad \forall s \in \mathbb{R}. \tag{5.3.28}$$

In the style of Lemma 5.1.4, we can find that this infimum is achieved at $\xi(s) \in \ell_1(\mathbb{R}_+)$, where

$$\xi_k = q_k^{(0)} \exp\left(\beta k \left[(\mu + s - a\delta^*)_- - s\mathbb{1}\{k \le J\}\right]\right), \quad k \in \mathbb{N}, \tag{5.3.29}$$

and $\delta^*(s)$ is given implicitly as the unique solution to the equation

$$\delta^* = \begin{cases} \sum_{j=1}^K j q_j^{(0)} \exp(\beta j(\mu - a\delta^*)) + \sum_{j=K+1}^\infty j q_j^{(0)} \exp(\beta j(\mu + s - a\delta^*)) & : \delta^* \ge \frac{\mu+s}{a}, \\ \sum_{j=1}^K j q_j^{(0)} \exp(-s\beta j) + \sum_{j=K+1}^\infty j q_j^{(0)} & : \delta^* \le \frac{\mu+s}{a}. \end{cases} \tag{5.3.30}$$

If we denote

$$\delta_K^* = \begin{cases} \sum_{j=1}^K j q_j^{(0)} \exp(\beta j(\mu - a\delta^*)) & : \delta^* \ge \frac{\mu+s}{a}, \\ \sum_{j=1}^K j q_j^{(0)} \exp(-s\beta j) & : \delta^* \le \frac{\mu+s}{a}, \end{cases} \tag{5.3.31}$$

then Lemma 5.2.2 tells us that

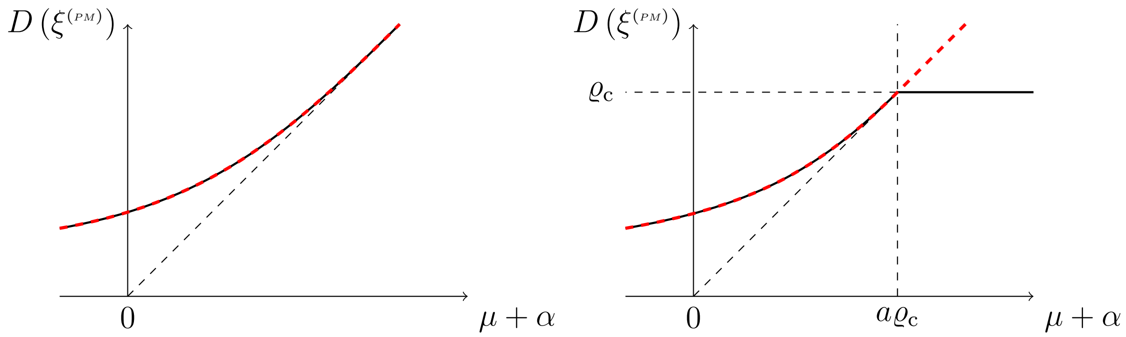
$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\lim_{N\to\infty} g_N^{(K)}(s)\right) = (\delta^* - \delta_K^*)(s) + \left(\frac{\mu+s}{a} - \delta^*(s)\right)_+ \tag{5.3.32}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\lim_{N\to\infty} g_N^{(K)}(s)\right)\Big|_{s=0} = \begin{cases} (\delta^* - \delta_K^*)(0) & : \mu \le a\varrho_{\mathrm{c}}, \\ \frac{\mu}{a} - \delta_K^*(0) & : \mu \ge a\varrho_{\mathrm{c}}, \end{cases} \tag{5.3.33}$$

$$\lim_{K\to\infty}\frac{\mathrm{d}}{\mathrm{d}s}\left(\lim_{N\to\infty} g_N^{(K)}(s)\right)\Big|_{s=0} = \left(\frac{\mu}{a} - \varrho_{\mathrm{c}}\right)_+. \tag{5.3.34}$$

□

The BEC phase transition for the PM model is established in various equivalent ways. In Theorem 5.3.3 it is shown that the excess particle density is carried by so-called loops of unbounded length. Alternatively, Proposition 5.2.10 and Proposition 5.2.11 establish the phase transition via the change of the pressure density relation. The advantage of our LDP approach is that the rate function has unique zero and not an approximating sequence of minimisers. This is due to the fact that we are using the lower semicontinuous regularisation of the energy proving the large deviation principle. A close inspection of Figure 5.5 reveals this. For $d \ge 3$, we know that $a\varrho_{\mathrm{c}} < \infty$, and thus the density of the zero of the rate function



(a) $d = 1, 2$

(b) $d \geq 3$

Figure 5.5: Total particle density of the zero of $I^{(PM)}_{\mu,\alpha}$. The limiting expected particle density (including the condensate) only differs for $\mu+\alpha > a\varrho_{\mathrm{c}}$, where it follows the dashed plot.

is constant for all $\mu + \alpha \geq a\varrho_{\mathrm{c}}$. In this region, the total particle density is the dashed line intersecting the point $(a\varrho_{\mathrm{c}}, \varrho_{\mathrm{c}})$. The so-called condensate density is then $(\frac{\mu+\alpha}{a} - \varrho_{\mathrm{c}})_+ = \Delta^{(PM)}(\beta, \mu, \alpha)$.

The following result shows that the condensate density has a limit for certain regimes of thermodynamic parameter β and μ and energy parameter a and b. More explicit expressions for the minimiser and condensate density in the critical regime are not yet available.

Theorem 5.3.4. *For $\beta > 0$, $\mu \in \mathbb{R}$, $\alpha \leq 0$, where the derivative is defined,*

$$\begin{aligned}\Delta^{(FCH)}(\beta,\mu,\alpha) &:= \lim_{K\to\infty}\lim_{N\to\infty}\mathbb{E}_{\nu^{(FCH)}_{N,\mu,\alpha}}\left[D - D_K\right] \\ &= -\lim_{K\to\infty}\frac{\mathrm{d}}{\mathrm{d}s}\left(\inf_{\ell_1}\left\{\frac{1}{\beta}I^{(FCH)}_{\mu+s,\alpha} + sD_K\right\} - p^{(FCH)}(\beta,\mu+s,\alpha)\right)\Bigg|_{s=0}.\end{aligned} \tag{5.3.35}$$

In particular, if the infimum is achieved by $\xi(s) \in C^1((-\epsilon,\epsilon) : \ell_1)$ for some $\epsilon > 0$, then

$$\Delta^{(FCH)}(\beta,\mu,\alpha) = \frac{a}{a-b}\left(\frac{\mu+\alpha}{a} - D(\xi(0))\right)_+ = \left(\frac{\partial}{\partial\mu}\left(H^{(FCH)}_{\mu+\alpha} - H^{(FCH)}_{\mu+\alpha,\mathrm{l.s.c.}}\right)\right)(\xi(0)). \tag{5.3.36}$$

Proof. The proof of (5.3.35) follows very similarly to the corresponding stage of the proof of Theorem 5.3.3. Note that the FCH version of (5.3.24) follows because $H^{(FCH)}_{\mu,\mathrm{l.s.c.}} \geq H^{(PM)}_{\mu,a-b,\mathrm{l.s.c.}}$ almost surely. The proof of (5.3.36) uses Lemma 5.2.2.

□

Remark 5.3.5. *The BEC phase transition for the FCH model is established as follows. Proposition 5.2.12 establishes a subcritical regime showing that for all* $\mu + \alpha \leq \tilde{\mu}$ *the pressure is smooth and its derivative gives the particle density with no condensation. In Proposition 5.2.13 we identify a regime for the inverse temperature where the pressure-density relation is broken. Depending on the density* $D(\xi(0))$*, for large enough* μ *the particle density in loops of unbounded length is not vanishing. In Figure 5.4 we can identify the regime* $\mu + \alpha \geq \overline{\mu}$ *when the density of the zero of the rate function is decreasing with* μ *such that the excess density is carried by loops of unbounded length. If we choose* $a = 2b$ *in* (5.3.36)*, we can recover the results in [Lew86] and [BLP88]. It shows in fact, that for increasing values of the coupling parameter* a*, the condensate density decreases. On the other hand, if the parameter for the counter energy term,* b*, is approaching* a*, the condensate density increases. This is due to the fact that with large counter terms the system distributes the physical particles in as few as possible different cycles lengths. To accommodate the particle density, the only way is to put them in infinitely long cycles. Our analysis actually shows that the BEC phase transition for FCH is more complex and requires further detailed study.* ◇

Chapter 6

Large Deviations for Mass-Split Bose Models

In Chapter 5 we saw how the full cycle HYL model can show a variety of interesting behaviours. In what follows we try to study a specific part of these by considering a partial cycle HYL (PCH) model. This is inspired by the version of the HYL considered by [BLP88], and our derivation of the thermodynamic pressure expands upon techniques used in their corresponding proof. That proof used a combination of large deviation techniques and direct combinatoric bounds. In Section 6.2, we apply these large deviation techniques to our PCH model. In particular, in the case where the free gas momentum eigenstates show no in-built spectral gap, we derive precisely the same thermodynamic pressure as [BLP88]. This is not a simple application of the standard Gärtner-Ellis Theorem ([DZ09, Theorem 2.3.6], for example) and Varadhan's Lemma, but requires a more hands-on approach in some important areas. To illustrate these differences, we derive a large deviation principle for the empirical total density in Section 6.1. In Section 6.3 we return to the momentum-space framework of [Lew86] and [BLP88] and see how far we can take the proof using only the large deviation techniques. We find that we can replicate their result up to a very similar (but not strictly weaker or stronger) condition. Then in Section 6.4 we consider the variational expressions we have for the thermodynamic pressures, and study the condensate behaviour.

In this chapter we will consider quite general parameters $q_k^{(N)}$ that also vary with N. In Section 6.1 we only require this first condition:

(A1): For all $u < 0$,

$$\left\{q_k^{(N)} \mathrm{e}^{uk}\right\}_{k\in\mathbb{N}} \xrightarrow{\ell_1} \left\{q_k \mathrm{e}^{uk}\right\}_{k\in\mathbb{N}}. \tag{6.0.1}$$

In Section 6.2, we introduce the non-empty subsets $A_N \subset \mathbb{N}$. We will require that if $\min A_N \ll |\Lambda_N|$, then there exist two sequences $r_N, r'_N \in A_N$ such that $r_N, r'_N \ll |\Lambda_N|$, $r_N \neq r'_N$ and they both satisfy

(A2):

$$\lim_{N\to\infty} q_{r_N}^{(N)} = 0, \tag{6.0.2}$$

(A3):

$$\lim_{N\to\infty} \frac{1}{r_N} \log q_{r_N}^{(N)} = 0. \tag{6.0.3}$$

In Appendix A we see that the parameters corresponding to the Dirichlet and periodic boundary conditions satisfy these conditions.

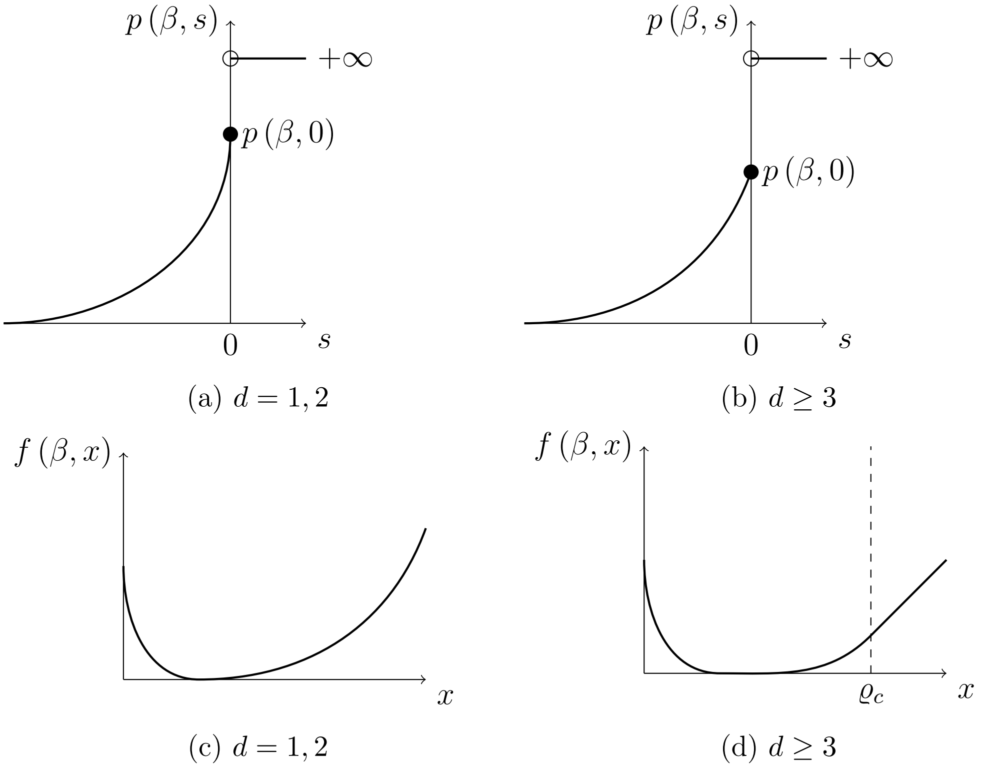
Also note that in this chapter, the inverse temperature $\beta > 0$ does little of interest that has not been encountered before. Therefore in our proofs we will set $\beta = 1$ and suppress it from the notation. Once we have these results, it is usually a matter of dimensional analysis to reintroduce it.

6.1 Total Density Large Deviations

In the mass-splitting arguments of the following sections, we will require large deviation principles not for the empirical cycle count but for the total particle densities associated with collections of cycle types. Deriving these is not a simple application of the Gärtner-Ellis Theorem or Contraction Principle as one might hope, but instead they have aspects that require a closer inspection. To demonstrate these difficulties and how to overcome them, we study here the empirical particle density.

Recall that the empirical particle density is defined as the unbounded linear transformation of the empirical cycle count:

$$\boldsymbol{\rho}_\Lambda = \sum_{k\in\mathbb{N}} k \boldsymbol{\lambda}_k^{(\Lambda)}. \tag{6.1.1}$$



(a) $d = 1, 2$ (b) $d \geq 3$

(c) $d = 1, 2$ (d) $d \geq 3$

Figure 6.1: Sketch of the ideal gas pressure p and free energy f. Note that p is steep for $d = 1, 2$, but not if $d \geq 3$, and this corresponds to f being affine for $x \geq \varrho_c$.

In turn we denote the reference distribution of $\boldsymbol{\rho}_{\Lambda_N}$ on $\mathbb{R}$ with chemical potential $\alpha < 0$ as $Q_{N,\alpha} = \mathsf{Q}_\alpha \circ \boldsymbol{\rho}_{\Lambda_N}^{-1}$. Note that for all $\omega \in \Omega$, we have $\boldsymbol{\rho}_{\Lambda_N}(\omega) < +\infty$ because $|\Lambda_N| < +\infty$, and so $Q_{N,\alpha}$ is a probability measure on $\mathbb{R}_+$.

We will also find it useful to define the functions

$$p(\beta, \alpha) = \begin{cases} \frac{1}{\beta} \sum_{k \in \mathbb{N}} q_k \mathrm{e}^{\beta \alpha k} & \alpha \leq 0 \\ +\infty & \alpha > 0, \end{cases} \tag{6.1.2}$$

$$f(\beta, x) = \sup_{s \in \mathbb{R}} \left\{ sx - p(\beta, s) \right\}. \tag{6.1.3}$$

Note that $p(\beta, 0) < +\infty$.

To perform our analysis, we will require an assumption on the densities $q_k^{(N)}$. We need them to approximate the empty boundary condition parameters in the following sense.

(A1): For all $u < 0$,

$$\left\{ q_k^{(N)} \mathrm{e}^{uk} \right\}_{k \in \mathbb{N}} \xrightarrow{\ell_1} \left\{ q_k \mathrm{e}^{uk} \right\}_{k \in \mathbb{N}}. \tag{6.1.4}$$

Remark 6.1.1. *Note that in Lemma A.2.2, we prove that the condition (A1) holds for* $\mathrm{bc} \in \{\mathrm{Dir}\,, \mathrm{per}\}$*. It also trivially holds for* $\mathrm{bc} = \varnothing$*.* $\diamond$

Lemma 6.1.2. *Suppose (A1) holds. Then for* $\alpha < 0$*, the logarithmic moment generating function for* $\boldsymbol{\rho}_N$ *is given by*

$$\mathcal{L}\left(s\right) \begin{cases} = \beta p\left(\beta, \frac{s}{\beta} + \alpha\right) - \beta p\left(\beta, \alpha\right) & : s < -\beta\alpha \\ \geq \beta p\left(\beta, 0\right) - \beta p\left(\beta, \alpha\right) & : s = -\beta\alpha \\ = +\infty & : s > -\beta\alpha. \end{cases} \tag{6.1.5}$$

This is given as a limit superior for $s = -\alpha$ *and a limit otherwise.*

The Legendre transform is given by

$$\mathcal{L}^*\left(x\right) = \beta f\left(\beta, x\right) - \beta\alpha x + \beta p\left(\beta, \alpha\right). \tag{6.1.6}$$

Proof. For fixed Λ the empirical particle density is a linear combination of independent Poisson random variables, and so it is a simple matter to calculate :

$$\mathcal{L}\left(s\right) = \limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log Q_{N,\alpha}\left[\mathrm{e}^{s|\Lambda_N|\boldsymbol{\rho}_{\Lambda_N}}\right] \tag{6.1.7}$$

$$= \limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \sum_{k\in\mathbb{N}} \log \mathbb{E}_{\nu_{N,\alpha}}\left[\mathrm{e}^{s|\Lambda_N|k\boldsymbol{\lambda}_k^{(\Lambda_N)}}\right] \tag{6.1.8}$$

$$= \limsup_{N\to\infty} \sum_{k\in\mathbb{N}} q_k^{(N)}\left(\mathrm{e}^{(s+\alpha)k} - \mathrm{e}^{\alpha k}\right). \tag{6.1.9}$$

Now because we have *(A1)*,

$$\lim_{N\to\infty} \sum_{k\in\mathbb{N}} q_k^{(N)} \mathrm{e}^{uk} = \sum_{k\in\mathbb{N}} q_k \mathrm{e}^{uk} = p\left(u\right), \qquad \text{for } u < 0. \tag{6.1.10}$$

Furthermore, the ℓ_1-convergence in *(A1)* implies pointwise convergence, and therefore for all $r \in \mathbb{N}$,

$$\lim_{N\to\infty} \sum_{k\in\mathbb{N}} q_k^{(N)} \mathrm{e}^{uk} \geq q_r \mathrm{e}^{ur}. \tag{6.1.11}$$

Since the left hand side is r-independent, for $u > 0$ we have,

$$\lim_{N\to\infty} \sum_{k\in\mathbb{N}} q_k^{(N)} \mathrm{e}^{uk} \geq \lim_{r\to\infty} q_r \mathrm{e}^{ur} = \infty. \tag{6.1.12}$$

To deal with $u = 0$, we note that $Q_{N,\alpha}\left[\mathrm{e}^{s|\Lambda_N|\boldsymbol{\rho}_{\Lambda_N}}\right]$ are non-decreasing in s, and therefore $\mathcal{L}(s)$ is also non-decreasing. This gives required the bound on the limit superior at $s = -\alpha$.

The Legendre transform is then given by

$$\begin{aligned} \mathcal{L}^*(x) &= \sup_{s\leq-\alpha} \{sx - \mathcal{L}(s)\} && (6.1.13)\\ &= \sup_{s\in\mathbb{R}} \{sx - p(s+\alpha)\} + p(\alpha) && (6.1.14)\\ &= f(x) - \alpha x + p(\alpha), && (6.1.15) \end{aligned}$$

as required.

□

Proposition 6.1.3. *Suppose (A1) holds. For $\alpha < 0$, the sequence of measures $\{Q_{N,\alpha}\}$ obeys a LDP on $\mathbb{R}$ with rate $|\Lambda_N|$ and good rate function*

$$I_\alpha(x) = \beta f(\beta, x) - \beta\alpha x + \beta p(\beta, \alpha). \tag{6.1.16}$$

Proof. We begin by attempting to apply Lemma 2.1.10. Lemma 6.1.2 gives the logarithmic moment generating function.

Now since $p(s+\alpha) < +\infty$ for $s+\alpha \leq 0$, the region $\operatorname{int}(\mathcal{D}_{\mathcal{L}}) = \{s \in \mathbb{R} : s < -\alpha\}$ and contains the origin. Therefore Lemma 2.1.10 immediately gives us the large deviation upper bound:

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log Q_{N,\alpha}(C) \leq -\inf_C \mathcal{L}^*, \qquad \forall C \subset \mathbb{R} \text{ closed}. \tag{6.1.17}$$

In spatial dimensions $d = 1, 2$, the function $\mathcal{L}$ is convex, essentially smooth, and lower semicontinuous. Hence Lemma 2.1.10 also gives us the large deviation lower bound in these cases. Unfortunately, in dimensions $d \geq 3$, the function $\mathcal{L}$ is not essentially smooth because it is not steep.

To prove the large deviation lower bound, it suffices to prove that for all $y \in \mathbb{R}$,

$$\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log Q_{N,\alpha}\left(B_y^\delta\right) \geq -\mathcal{L}^*\left(y\right), \tag{6.1.18}$$

where B_y^δ is the open ball of radius δ centred upon y. First note that this is trivially true for $y < 0$. The argument for $y \in [0, \varrho_c)$ follows the standard line (see for example, [DZ09]). Fix $y \in [0, \varrho_c)$, so that y is an exposed point of $\mathcal{L}^*$ and the exposing hyperplane η exists uniquely with $\eta \in \operatorname{int}(\mathcal{D}_{\mathcal{L}})$. Then we can define the associated probability measures $\tilde{Q}_{N,\alpha}$ by their Radon-Nikodym derivative

$$\frac{\mathrm{d}\tilde{Q}_{N,\alpha}}{\mathrm{d}Q_{N,\alpha}}\left(z\right) = \exp\left(|\Lambda_N|\eta z - \mathcal{L}_N\left(|\Lambda_N|\eta\right)\right). \tag{6.1.19}$$

Note that this definition fixes $\tilde{Q}_{N,\alpha} = Q_{N,\alpha+\eta}$. From the definition of $\tilde{Q}_{N,\alpha}$, it follows that

$$\frac{1}{|\Lambda_N|}\log Q_{N,\alpha}\left(B_y^\delta\right) = \frac{1}{|\Lambda_N|}\mathcal{L}_N\left(|\Lambda_N|\eta\right) - \eta y + \frac{1}{|\Lambda_N|}\log\int_{z\in B_y^\delta}\mathrm{e}^{|\Lambda_N|\eta(y-z)}\mathrm{d}\tilde{Q}_{N,\alpha} \tag{6.1.20}$$

$$\geq \frac{1}{|\Lambda_N|}\mathcal{L}_N\left(|\Lambda_N|\eta\right) - \eta y - |\eta|\delta + \frac{1}{|\Lambda_N|}\log\tilde{Q}_{N,\alpha}\left(B_y^\delta\right). \tag{6.1.21}$$

Now taking the appropriate limits gives us the bound

$$\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log Q_{N,\alpha}\left(B_y^\delta\right) \geq \mathcal{L}\left(\eta\right) - \eta y + \lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log\tilde{Q}_{N,\alpha}\left(B_y^\delta\right) \tag{6.1.22}$$

$$\geq -\mathcal{L}^*\left(y\right) + \lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log\tilde{Q}_{N,\alpha}\left(B_y^\delta\right). \tag{6.1.23}$$

From the construction of $\tilde{Q}_{N,\alpha}$, we can find the corresponding logarithmic moment generating function and the Legendre transform:

$$\tilde{\mathcal{L}}\left(s\right) = \mathcal{L}\left(s+\eta\right) - \mathcal{L}\left(\eta\right) \tag{6.1.24}$$

$$\tilde{\mathcal{L}}^*\left(x\right) = \mathcal{L}^*\left(x\right) - \eta x + \mathcal{L}\left(\eta\right). \tag{6.1.25}$$

This allows us to apply Lemma 2.1.10 to get the large deviation upper bound for $\tilde{Q}_{N,\alpha}$ and apply it to the closed set $\left(B_y^\delta\right)^c$.

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \tilde{Q}_{N,\alpha}\left(\left(B_y^\delta\right)^c\right) \leq - \inf_{x\in\left(B_y^\delta\right)^c} \tilde{\mathcal{L}}^*(x) = -\tilde{\mathcal{L}}^*(x_0), \tag{6.1.26}$$

for some $x_0 \neq y$. The equality follows from $\tilde{\mathcal{L}}^*$ having compact level sets. Since $\mathcal{L}^*(y) \geq \eta y - \mathcal{L}(\eta)$ and y is an exposed point with exposing hyperplane η,

$$\tilde{\mathcal{L}}^*(x_0) = \mathcal{L}^*(x_0) - \eta x_0 + \mathcal{L}(\eta) \geq \mathcal{L}^*(x_0) - \eta x_0 - \left(\mathcal{L}^*(y) - \eta y\right) > 0. \tag{6.1.27}$$

Combining (6.1.26) and (6.1.27) gives us $\tilde{Q}_{N,\alpha}\left(\left(B_y^\delta\right)^c\right) \to 0$ and $\tilde{Q}_{N,\alpha}\left(B_y^\delta\right) \to 1$ for all $\delta > 0$. This proves (6.1.18) for $y < \varrho_c$.

For $y \geq \varrho_c$ we begin similarly. There is no exposing hyperplane now, but we can construct $\tilde{Q}_{N,\alpha}$ with the Radon-Nikodym derivatives

$$\frac{\mathrm{d}\tilde{Q}_{N,\alpha}}{\mathrm{d}Q_{N,\alpha}}(z) = \exp\left(|\Lambda_N|(\eta-\alpha)z - \mathcal{L}_N\left(|\Lambda_N|(\eta-\alpha)\right)\right). \tag{6.1.28}$$

where $\eta < 0$. In particular, note that $\tilde{Q}_{N,\alpha} = Q_{N,\eta}$ and that $\mathcal{L}(\eta-\alpha)$ exists as a limit. The argument then proceeds as before until we reach

$$\lim_{\delta\to 0}\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log Q_{N,\alpha}\left(B_y^\delta\right) \geq -\mathcal{L}^*(y) + \lim_{\delta\to 0}\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \tilde{Q}_{N,\alpha}\left(B_y^\delta\right). \tag{6.1.29}$$

We now claim that

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \tilde{Q}_{N,\alpha}\left(B_y^\delta\right) \geq \eta\left(y-\varrho_c\right), \tag{6.1.30}$$

and by choosing η arbitrarily small we have proven (6.1.18).

Lemma 6.1.4. *Let $\eta < 0$. Suppose $\boldsymbol{X}^{(N)} = \sum_{k\in\mathbb{N}} kX_k^{(N)}$, where $X_k^{(N)}$ are independent Poisson random variables with means $Nq_k^{(N)}\mathrm{e}^{\eta k}$ respectively. Let*

$$\varrho_c^\eta = \lim_{N\to\infty} \sum_{k\in\mathbb{N}} kq_k^{(N)}\mathrm{e}^{\eta k} \tag{6.1.31}$$

and $\delta > 0$. Then for $y > \varrho_c^\eta$,

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\frac{1}{N}\boldsymbol{X}^{(N)} \in B_y^\delta\right) \geq \eta\left(y - \varrho_c^\eta\right). \tag{6.1.32}$$

Proof. First fix $r \in \mathbb{N}$. Then we restrict our event to a particular partition of mass between the $k < r$, $k = r$ and $k > r$ states. Denote $\varrho_{N,r}^< = \sum_{k<r} k q_k^{(N)} \mathrm{e}^{\eta k}$, so

$$\mathbb{P}\left(\frac{1}{N}\boldsymbol{X}^{(N)} \in B_y^\delta\right) \geq \mathbb{P}\left(\frac{1}{N}\sum_{k<r} kX_k^{(N)} \in B_{\varrho_{N,r}^<}^{\delta/2}\right)\mathbb{P}\left(\frac{1}{N} rX_r^{(N)} \in B_{y-\varrho_{N,r}^<}^{\delta/2}\right) \times \mathbb{P}\left(\sum_{k>r}\left|X_k^{(N)}\right| = 0\right). \tag{6.1.33}$$

Now due to being Poisson random variables,

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\sum_{k>r}\left|X_k^{(N)}\right| = 0\right) = -\lim_{N\to\infty}\sum_{k>r} q_k^{(N)}\mathrm{e}^{\eta k} = -\sum_{k>r} q_k \mathrm{e}^{\eta k}, \tag{6.1.34}$$

where the limit holds from condition *(A1)*. Furthermore, the Poisson structure and the linearity of expectation and variance implies that $\sum_{k<r} kX_k^{(N)}$ has expectation and variance $N\varrho_{N,r}^<$. Chebyshev's inequality then implies that

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{N}\sum_{k<r} kX_k^{(N)} \in B_{\varrho_{N,r}^<}^{\delta/2}\right) \geq 1 - \lim_{N\to\infty} \frac{4\varrho_{N,r}^<}{N\delta^2} = 1. \tag{6.1.35}$$

For the last factor, we choose z such that both $z \in B_{y-\varrho_{N,r}^<}^{\delta/2}$ and $\frac{r}{N}\left\lfloor \frac{N}{r} z\right\rfloor \in B_{y-\varrho_{N,r}^<}^{\delta/2}$ eventually. Then by applying Stirling's formula with $N \to \infty$ we get

$$\frac{1}{N}\log \mathbb{P}\left(X_r^{(N)} = \left\lfloor \frac{N}{r} z\right\rfloor\right) = -\frac{1}{N}\left\lfloor \frac{N}{r}z\right\rfloor\left(\log \frac{\left\lfloor \frac{N}{r}z\right\rfloor}{Nq_r^{(N)}\mathrm{e}^{\eta r}} - 1\right) - q_r^{(N)}\mathrm{e}^{\eta r} + O\left(\frac{1}{N}\log N\right) \tag{6.1.36}$$

$$= -\frac{z}{r}\left(\log\frac{z}{r} - \log q_r - 1\right) + \eta z - q_r \mathrm{e}^{\eta r} + o(1). \tag{6.1.37}$$

In particular, choosing $z = y - \varrho_c^\eta$ satisfies our conditions on z for r sufficiently large.

In summary, for any $r \in \mathbb{N}$ sufficiently large we have

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\frac{1}{N}\boldsymbol{X}^{(N)} \in B_y^\delta\right) \geq -\frac{y-\varrho_c^\eta}{r}\left(\log \frac{y-\varrho_c^\eta}{r} - \log q_r - 1\right) + \eta\left(y-\varrho_c^\eta\right) - \sum_{k\geq r} q_k \mathrm{e}^{\eta k}. \tag{6.1.38}$$

Since the left hand side is r-independent, we can take $r \to \infty$ and get the required result.

□

As described above, this lemma proves (6.1.18).

□

As an accompaniment to Lemma 6.1.4, the following lemma shows that even with $\eta = 0$, collections of such events do decay - just not at a $|\Lambda_N|$-exponential rate.

Lemma 6.1.5. *Suppose $\boldsymbol{X}^{(N)} = \sum_{k\in\mathbb{N}} kX_k^{(N)}$, where $X_k^{(N)}$ are independent Poisson random variables with means Nq_k respectively, and $\varrho_c = \sum_{k\in\mathbb{N}} kq_k < \infty$. Then for $\delta > 0$,*

$$\mathbb{P}\left(\boldsymbol{X}^{(N)} \geq N\left(\varrho_c + \delta\right)\right) = O\left(\frac{1}{N}\right). \tag{6.1.39}$$

Proof. As Poisson random variables, $\mathrm{Var}\left[X_k^{(N)}\right] = Nq_k$. By linearity of expectation and variance,

$$\mathbb{E}\left[\boldsymbol{X}^{(N)}\right] = N\varrho_c, \qquad \mathrm{Var}\left[\boldsymbol{X}^{(N)}\right] = N\varrho_c. \tag{6.1.40}$$

Then we increase the size of the target set and apply Chebyshev's inequality:

$$\begin{aligned} \mathbb{P}\left(\boldsymbol{X}^{(N)} \geq N\left(\varrho_c + \delta\right)\right) &\leq \mathbb{P}\left(\left|\boldsymbol{X}^{(N)} - N\varrho_c\right| \geq N\delta\right) && (6.1.41)\\ &\leq \frac{1}{N}\frac{\varrho_c}{\delta^2}. && (6.1.42)\end{aligned}$$

□

6.2 Partial Cycle HYL Model Large Deviations

The aim of this section is to derive a variational expression for the thermodynamic pressure of the model. The true HYL model is diagonal in the momentum eigenstate occupation numbers, and the thermodynamic pressure for a very similar model was found by Lewis in [Lew86]. Furthermore, Lewis' model only had the counter-term affect momentum eigenstates below some eigenvalue number cutoff that approached infinity. In contract, here we replace the momentum eigenstate occupation numbers with the cycle-type occupation numbers, and have the counter term affect cycles with time horizon *greater than* some cutoff that is taken to infinity. The work in this section is related to the author's work appearing in [AD19].

In order to have exponential tightness for our underlying model, we will need to take the base ideal gas model with chemical potential $\alpha < 0$. Then we add the interaction energy via the tilt

$$H_N^{(\text{PCH})}(\omega) = -(\mu-\alpha)\, N_{\Lambda_N}^{(\ell)}(\omega) + \frac{a}{2|\Lambda_N|}\left(N_{\Lambda_N}^{(\ell)}(\omega)\right)^2 - \frac{b}{2|\Lambda_N|}\sum_{k\in A_N}\left(k\mathcal{N}_{\Lambda_N,k}(\omega)\right)^2, \tag{6.2.1}$$

where $a > 0$, $b < a$, $\mu \in \mathbb{R}$, and the subsets $A_N \subset \mathbb{N}$ satisfy $\min A_N \to +\infty$. As the case with "eventually $A_N = \varnothing$" coincides with the Particle Mean Field model, let us suppose A_N is always non-empty. We aim to find an expression for the thermodynamic pressure:

$$p^{(\text{PCH})}(\beta,\mu,\alpha) = p(\beta,\alpha) + \lim_{N\to\infty}\frac{1}{\beta|\Lambda_N|}\log \mathbb{E}_{\nu_{N,\alpha}}\left[\mathrm{e}^{-\beta H_{\Lambda_N}^{(\text{PCH})}}\right]. \tag{6.2.2}$$

Since the energetic chemical potential has been set to $\mu-\alpha$ and we have the addition of $p(\beta,\alpha)$, these cancel out the α-dependent parts of $\mathbb{E}_{\nu_{N,\alpha}}$ and so we have that $p^{(\text{PCH})}(\beta,\mu,\alpha)$ is α-independent. We will therefore suppress the α from this notation.

Remark 6.2.1. *We shall find it useful to have the following expression for the thermodynamic free energy of the ideal Bose gas:*

$$f(\beta,x) = \sup_{s\in\mathbb{R}}\left\{sx - \frac{1}{\beta}\sum_{j\in\mathbb{N}} q_j \mathrm{e}^{\beta s j}\right\} = \sup_{s\in\mathbb{R}}\left\{sx - p(\beta,s)\right\}. \tag{6.2.3}$$

Due to the differentiability of $p(\beta, s)$ for $s < 0$ and the divergence to $+\infty$ for $s > 0$, it is simple to see that this expression is equal to the version found in [Ada08, Th. 2.4]. $\diamond$

Recall that we have made two extra assumptions on the double sequence $\left\{q_k^{(N)}\right\}_{N,k\in\mathbb{N}}$. We require that if $\min A_N \ll |\Lambda_N|$, then there exist two sequences $r_N, r'_N \in A_N$ such that $r_N, r'_N \ll |\Lambda_N|$, $r_N \neq r'_N$ and they both satisfy

(A2):

$$\lim_{N\to\infty} q_{r_N}^{(N)} = 0, \tag{6.2.4}$$

(A3):

$$\lim_{N\to\infty} \frac{1}{r_N} \log q_{r_N}^{(N)} = 0. \tag{6.2.5}$$

We consider a 2-splitting of the total particle density. Let $\left(M_1^{(2,N)}, M_2^{(2,N)}\right)$ be random variables, where

$$M_1^{(2,N)}(\omega) = \sum_{j\in\mathbb{N}\setminus A_N} j\boldsymbol{\lambda}_j^{(N)}(\omega) \qquad M_2^{(2,N)}(\omega) = \sum_{j\in A_N} j\boldsymbol{\lambda}_j^{(N)}(\omega), \tag{6.2.6}$$

and define the 2-splitting maps

$$\pi_N^{(2)} : \Omega \to \mathbb{R}_+^2, \qquad \omega \mapsto \left(M_1^{(2,N)}, M_2^{(2,N)}\right). \tag{6.2.7}$$

Now let $Q_{N,\alpha}^{(2)}$ be the 2-splitting probability measures on $\mathbb{R}_+^2$ induced by the non-interacting reference measures $\mathsf{Q}_{N,\alpha}$:

$$Q_{N,\alpha}^{(2)} = \mathsf{Q}_{N,\alpha} \circ \left(\pi_N^{(2)}\right)^{-1}. \tag{6.2.8}$$

Similarly, let $Q_{N,\alpha}^{(2,PCH)}$ be the 2-splitting probability measures on $\mathbb{R}_+^2$ induced by the PCH reference measures $\mathsf{Q}_{N,\mu}^{(PCH)}$:

$$Q_{N,\mu}^{(2,PCH)} = \mathsf{Q}_{N,\mu}^{(PCH)} \circ \left(\pi_N^{(2)}\right)^{-1}. \tag{6.2.9}$$

Theorem 6.2.2. *Let $\beta > 0$, $\alpha < 0$, $\mu \in \mathbb{R}$, $a > 0$, $b < a$ and non-empty $A_N \subset \mathbb{N}$*

such that $\min A_N \to +\infty$. *Suppose (A1) holds.*

- *If* $\min A_N \ll |\Lambda_N|$, *and (A2) and (A3) hold, then the sequence of measures* $\left\{Q_{N,\mu}^{(2,PCH)}\right\}_{N\in\mathbb{N}}$ *obeys a LDP on* $\mathbb{R}_+^2$ *with rate* $|\Lambda_N|$ *and good rate function*

$$\mathcal{J}^{(PCH)}(x,y) = \beta f(\beta,x) - \beta\mu(x+y) + \frac{\beta a}{2}(x+y)^2 - \frac{\beta b}{2}y^2 + \beta p^{(PCH)}(\beta,\mu), \tag{6.2.10}$$

where $p^{(PCH)}(\beta,\mu)$ *is the thermodynamic pressure:*

$$p^{(PCH)}(\beta,\mu) = \sup_{x,y\geq 0}\left\{\mu(x+y) - \frac{a}{2}(x+y)^2 + \frac{b}{2}y^2 - f(\beta,x)\right\} \tag{6.2.11}$$

$$= \sup_{x\geq 0}\left\{\mu x - \frac{a}{2}x^2 + \frac{(\mu - ax)_+^2}{2(a-b)} - f(\beta,x)\right\}. \tag{6.2.12}$$

- *If* $\min A_N \gg |\Lambda_N|$, *the sequence of measures* $\left\{Q_{N,\mu}^{(2,PCH)}\right\}_{N\in\mathbb{N}}$ *obeys a LDP on* $\mathbb{R}_+^2$ *with rate* $|\Lambda_N|$ *and good rate function*

$$\mathcal{J}^{(PCH)}(x,y) = \begin{cases} \beta f(\beta,x) - \beta\mu x + \frac{\beta a}{2}x^2 + \beta p^{(PM)}(\beta,\mu) & : y = 0 \\ +\infty & : y > 0, \end{cases} \tag{6.2.13}$$

where $p^{(PM)}(\beta,\mu)$ *is the PM thermodynamic pressure:*

$$p^{(PM)}(\beta,\mu) = \sup_{x\geq 0}\left\{\mu x - \frac{a}{2}x^2 - f(\beta,x)\right\}. \tag{6.2.14}$$

Remark 6.2.3. *The* $\min A_N \gg |\Lambda_N|$ *case is different because this condition essentially removes the HYL counter-terms. For finite* N*, the two smallest permitted values of the random variable* $M_2^{(2,N)}$ *are* 0 *and* $\frac{\min A_N}{|\Lambda_N|}$. *This case then forces all non-zero permitted values to diverge to infinity. These are therefore naturally unlikely, and the* A_N *states aren't allowed to support any particle mass. Therefore the PCH interaction energy is essentially equal to the PM model interaction in this case.* $\diamond$

Remark 6.2.4. *We do not address the case whereby there exist* $c_1, c_2 \in (0,+\infty)$ *such that* $\frac{\min A_N}{|\Lambda_N|} \in (c_1, c_2)$ *eventually. This is because this case can depend very delicately on the combinatoric considerations of which values* $M_2^{(2,N)}$ *can take. In* *general the limits of these values can be very complex and we defer for future study.* $\diamond$

Remark 6.2.5. *Note that the topology for which we have the LDP in Theorem 6.2.2 is the standard Euclidean topology on $\mathbb{R}_+^2$, rather than the ℓ_1-topology or topology of local convergence we have considered previously. In these previous cases, we had to work hardest to deal with the lack of continuity in the Hamiltonian whilst proving the LDP for the base measure was relatively easy or already performed. However, we not find that the main task is to prove the LDP for base measure and once we have done this it is easy to bound the PCH Hamiltonian with continuous Hamiltonians.* $\diamond$

We first suppose that $\min A_N \ll |\Lambda_N|$ and $b \geq 0$. For $b < 0$ our proof will be identical but with the roles of $\overline{H}$ and $\underline{H}$ (see (6.2.52) and (6.2.53)) reversed. We shall return to the $\min A_N \gg |\Lambda_N|$ case at the end where we supplement our prior discussion with a "by hands" segment that bridges the gap.

We begin by splitting the random variable $M_2^{(2,N)}$ in two. We extract the particle density associated with a particular cycle length, specifically the r_N cycles identified in conditions *(A2)* and *(A3)*. Let $\left(M_1^{(3,N)}, M_2^{(3,N)}, M_3^{(3,N)}\right)$ be random variables, where

$$M_1^{(3,N)}(\omega) = \sum_{j \in \mathbb{N} \setminus A_N} j \boldsymbol{\lambda}_j^{(N)}(\omega) \tag{6.2.15}$$

$$M_2^{(3,N)}(\omega) = r_N \boldsymbol{\lambda}_{r_N}^{(N)}(\omega) \tag{6.2.16}$$

$$M_3^{(3,N)}(\omega) = \sum_{j \in A_N \setminus \{r_N\}} j \boldsymbol{\lambda}_j^{(N)}(\omega) \tag{6.2.17}$$

and define the maps

$$\pi_N^{(3)} : \Omega \to \mathbb{R}_+^3, \qquad \omega \mapsto \left(M_1^{(3,N)}, M_2^{(3,N)}, M_3^{(3,N)}\right). \tag{6.2.18}$$

Now let $Q_{N,\alpha}^{(3)}$ be the 3-splitting probability measures induced by the non-interacting reference measures $\mathtt{Q}_{N,\alpha}$:

$$Q_{N,\alpha}^{(3)} = \mathtt{Q}_{N,\alpha} \circ \left(\pi_N^{(3)}\right)^{-1}. \tag{6.2.19}$$

Lemma 6.2.6. *Suppose (A1) and (A3) hold. Then the limiting logarithmic moment generating function for* $\left\{Q^{(3)}_{N,\alpha}\right\}_{N\in\mathbb{N}}$ *is given by*

$$\mathcal{L}(s,t,u)\begin{cases} = \beta p\left(\beta, \frac{s}{\beta}+\alpha\right) - \beta p(\beta,\alpha) & : s < -\beta\alpha \text{ and } t < -\beta\alpha \text{ and } u < -\beta\alpha \\ = +\infty & : s > -\beta\alpha \text{ or } t > -\beta\alpha \text{ or } u > -\beta\alpha \\ \geq \beta p\left(\beta, \frac{s}{\beta}+\alpha\right) - \beta p(\beta,\alpha) & : \text{otherwise,} \end{cases} \tag{6.2.20}$$

and is a limit for the first two cases.

The Legendre-Fenchel transform is given by

$$\mathcal{L}^*(x,y,z) = \beta f(\beta,x) - \beta\alpha(x+y+z) + \beta p(\beta,\alpha). \tag{6.2.21}$$

Proof. From $\left\{|\Lambda_N|\boldsymbol{\lambda}^{(N)}_j\right\}_{j\in\mathbb{N}}$ being independent Poisson random variables,

$$\mathcal{L}(s,t,u) := \limsup_{N\to\infty} \frac{1}{|\Lambda_N|}\log Q^{(3)}_{N,\alpha}\left[\mathrm{e}^{|\Lambda_N|\left(sM_1^{(3,N)}+tM_2^{(3,N)}+uM_3^{(3,N)}\right)}\right] \tag{6.2.22}$$

$$\begin{aligned} = \limsup_{N\to\infty}\Bigg(&\sum_{k\in\mathbb{N}\setminus A_N} \frac{1}{|\Lambda_N|}\log\mathbb{E}_{\nu_{N,\alpha}}\left[\mathrm{e}^{|\Lambda_N|sk\boldsymbol{\lambda}^{(N)}_k}\right] \\ &+\frac{1}{|\Lambda_N|}\log\mathbb{E}_{\nu_{N,\alpha}}\left[\mathrm{e}^{|\Lambda_N|tr_N\boldsymbol{\lambda}^{(N)}_{r_N}}\right] + \sum_{k\in A_N\setminus\{r_N\}}\frac{1}{|\Lambda_N|}\log\mathbb{E}_{\nu_{N,\alpha}}\left[\mathrm{e}^{|\Lambda_N|uk\boldsymbol{\lambda}^{(N)}_k}\right]\Bigg) \end{aligned} \tag{6.2.23}$$

$$\begin{aligned} = \limsup_{N\to\infty}\Bigg(&\sum_{k\in\mathbb{N}\setminus A_N} q^{(N)}_k\mathrm{e}^{(s+\alpha)k} + q^{(N)}_{r_N}\mathrm{e}^{(t+\alpha)r_N} + \sum_{k\in A_N\setminus\{r_N\}} q^{(N)}_k\mathrm{e}^{(u+\alpha)k} \\ &-\sum_{k\in\mathbb{N}} q^{(N)}_k\mathrm{e}^{\alpha k}\Bigg). \end{aligned} \tag{6.2.24}$$

Because we have the ℓ_1-convergence in *(A1)*, we have the required limit for $s < -\alpha$, $t < -\alpha$ and $u < -\alpha$. From condition *(A3)*, we know that there are sequences $r_N \in A_N$ and $r'_N \in A_N \setminus \{r_N\}$ such that $q^{(N)}_{r_N}\mathrm{e}^{vr_N} \to \infty$ and $q^{(N)}_{r'_N}\mathrm{e}^{vr'_N} \to \infty$ for all

$v > 0$. This also implies

$$\lim_{N\to\infty} \sum_{k\in A_N\setminus\{r_N\}} q_k^{(N)} \mathrm{e}^{vk} \geq \lim_{N\to\infty} q_{r'_N}^{(N)} \mathrm{e}^{vr'_N} = +\infty, \tag{6.2.25}$$

for $v > 0$. The corresponding result for $\mathbb{N} \setminus A_N$ follows similarly whilst considering k growing sufficiently slowly.

To deal with $u = 0$, we note that $Q_{N,\alpha}^{(3)} \left[\mathrm{e}^{|\Lambda_N|\left(sM_1^{(3,N)}+tM_2^{(3,N)}+uM_3^{(3,N)}\right)} \right]$ are non-decreasing in s, t and u, and therefore $\mathcal{L}(s,t,u)$ is also non-decreasing.

The Legendre-Fenchel transform is then

$$\begin{aligned} \mathcal{L}^*(x,y,z) &:= \sup_{s,t,u\in\mathbb{R}} \{sx + ty + uz - \mathcal{L}(s,t,u)\} && (6.2.26) \\ &= \sup_{s<-\alpha} \{sx - p(s+\alpha)\} + \sup_{t<-\alpha} \{ty\} + \sup_{u<-\alpha} \{uz\} + p(\alpha) && (6.2.27) \\ &= \sup_{s<0} \{sx - p(s)\} - \alpha(x+y+z) + p(\alpha), && (6.2.28) \end{aligned}$$

as required.

□

Lemma 6.2.7. *Suppose (A1) holds and $\alpha < 0$. We define the rate function $\mathcal{J}_\alpha(x,y,z)$ thus:*

- *if* $\min A_N \ll |\Lambda_N|$, *and (A2) and (A3) hold,*

$$\mathcal{J}_\alpha(x,y,z) = \beta f(\beta,x) - \beta\alpha(x+y+z) + \beta p(\beta,\alpha). \tag{6.2.29}$$

- *if* $\min A_N \gg |\Lambda_N|$,

$$\mathcal{J}_\alpha(x,y,z) = \begin{cases} \beta f(\beta,x) - \beta\alpha x + \beta p(\beta,\alpha) & : y = 0 \text{ and } z = 0 \\ +\infty & : y > 0 \text{ or } z > 0. \end{cases} \tag{6.2.30}$$

Then in these cases the sequence of measures $\left\{Q_{N,\alpha}^{(3)}\right\}_{N\in\mathbb{N}}$ obeys a LDP on $\mathbb{R}_+^3$ with rate $|\Lambda_N|$ and good rate function $\mathcal{J}_\alpha(x,y,z)$.

PROOF. We first address $\min A_N \ll |\Lambda_N|$. Having the logarithmic moment generating function and its Legendre-Fenchel transform from Lemma 6.2.6, one would hope that we would apply the Gärtner-Ellis Theorem. Indeed, $0 \in \operatorname{int}(\mathcal{D}_{\mathcal{L}})$ and so we are given the large deviation limit superior bound. However, as in the case of Proposition 6.1.3, $\mathcal{L}$ is not steep (now for any $d \geq 1$), and so we do not have a direct application of the lim inf bound. In fact, because $\mathcal{L}^*(x, y, z)$ is linear in y and z, it has no exposing points at all!

We resolve this issue in a similar way to Proposition 6.1.3. First let us use the independence of $M_1^{(3,N)}$, $M_2^{(3,N)}$ and $M_3^{(3,N)}$ to get three clearer problems. Say $O \subset \mathbb{R}_+^3$ is our given open set. Then let $w = (w_1, w_2, w_3) \in O$ be arbitrary and choose $\delta > 0$ such that $(w_1 - \delta, w_1 + \delta) \times (w_2 - \delta, w_2 + \delta) \times (w_3 - \delta, w_3 + \delta) \subset O$. If we denote with $Q_i^{(N)}$ the marginal distribution of $M_i^{(N)}$, then by independence

$$Q_{N,\alpha}^{(3)}(O) \geq Q_1^{(N)}\left(B_{w_1}^{\delta}\right) Q_2^{(N)}\left(B_{w_2}^{\delta}\right) Q_3^{(N)}\left(B_{w_3}^{\delta}\right) \tag{6.2.31}$$

and we can consider the large deviations of each marginal separately.

The large deviation bound on $Q_1^{(N)}\left(B_{w_1}^{\delta}\right)$ is proven in essentially the same way as for Proposition 6.1.3. The arguments of Lemma 6.2.6 repeat to give the logarithmic moment generating function of $Q_1^{(N)}$ and its Legendre-Fenchel transform to be

$$\mathcal{L}_1(s) \begin{cases} = p(s+\alpha) - p(\alpha) & : s < -\alpha \\ \geq p(s+\alpha) - p(\alpha) & : s = -\alpha \\ = +\infty & : s > -\alpha, \end{cases} \tag{6.2.32}$$

$$\mathcal{L}_1^*(x) = f(x) - \alpha x + p(\alpha). \tag{6.2.33}$$

For $w_1 < \varrho_c$ the argument is the standard Gärtner-Ellis one described before because w_1 will be an exposed point of $\mathcal{L}_1^*$. If $w_1 \geq \varrho_c$, then we can proceed as before using Lemma 6.1.4 - the lemma and proof need only superficial changes.

For $Q_2^{(N)}$, we find that

$$\mathcal{L}_2\left(t\right)\begin{cases}=0 & : t<-\alpha\\ \geq 0 & : t=-\alpha\\ =+\infty & : t>-\alpha.\end{cases} \tag{6.2.34}$$

$$\mathcal{L}_2^*\left(y\right)=-\alpha y. \tag{6.2.35}$$

Now our only case is analogous to the $w_1 \geq \varrho_c$ case for $Q_1^{(N)}$, but we are unable to fix r because eventually $r < \min A_N$.

Lemma 6.2.8. *Consider a sequence of sets $E_N \subset \mathbb{N}$. Let $\eta < 0$, $1 \ll \min E_N \ll N$, suppose that (A1) holds, and (A2) and (A3) hold for a sequence of points in E_N. Also let $\boldsymbol{X}^{(N)} = \sum_{k\in A_N} kX_k^{(N)}$, where $X_k^{(N)}$ are independent Poisson random variables with means $Nq_k^{(N)}\mathrm{e}^{\eta k}$ respectively. Then for $w \geq 0$ and $\delta > 0$,*

$$\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\frac{1}{N}\boldsymbol{X}^{(N)}\in B_w^\delta\right)\geq \eta w. \tag{6.2.36}$$

Proof. Let $r_N \in E_N$ be the sequence identified by conditions *(A2)* and *(A3)*. In particular, $r_N \ll N$. Therefore if $\zeta \in B_w^\delta$ then

$$\frac{r_N}{N}\left\lfloor\frac{N}{r_N}\zeta\right\rfloor=\zeta+O\left(\frac{r_N}{N}\right) \tag{6.2.37}$$

is in B_w^δ eventually.

Now let us restrict the event $\frac{1}{N}\boldsymbol{X}^{(N)} \in B_w^\delta$ to $X_{r_N}^{(N)} = \left\lfloor\frac{N}{r_N}w\right\rfloor$ and $X_k^{(N)} = 0$ for all $k \in E_N \setminus \{r_N\}$:

$$\mathbb{P}\left(\frac{1}{N}\boldsymbol{X}^{(N)}\in B_w^\delta\right)\geq\mathbb{P}\left(X_k^{(N)}=0,\forall k\in E_N\setminus\{r_N\}\right)\mathbb{P}\left(X_{r_N}^{(N)}=\left\lfloor\frac{N}{r_N}w\right\rfloor\right). \tag{6.2.38}$$

Now

$$\frac{1}{N}\log\mathbb{P}\left(X_k^{(N)}=0,\forall k\in A_N\setminus\{r_N\}\right)=-\sum_{k\in E_N\setminus\{r_N\}}q_k^{(N)}e^{\eta k}. \tag{6.2.39}$$

Because we have $\min E_N \to 0$ and *(A1)*, this right hand side vanishes in the limit. By applying Stirling's approximation and conditions *(A2)* and *(A3)*,

$$\frac{1}{N}\log\mathbb{P}\left(X^{(N)}_{r_N}=\left\lfloor\frac{N}{r_N}w\right\rfloor\right)=-\frac{1}{N}\left\lfloor\frac{N}{r_N}w\right\rfloor\left(\log\frac{\left\lfloor\frac{r_N}{N}w\right\rfloor}{Nq^{(N)}_{r_N}\mathrm{e}^{\eta r_N}}-1\right)$$
$$-q^{(N)}_{r_N}\mathrm{e}^{\eta r_N}+O\left(\frac{1}{N}\log\frac{N}{r_N}\right)\tag{6.2.40}$$
$$=-\frac{w}{r_N}\left(\log\frac{w}{r_N}-\log q^{(N)}_{r_N}-1\right)+\eta w-q^{(N)}_{r_N}\mathrm{e}^{\eta r_N}+o\left(1\right)\tag{6.2.41}$$
$$=\eta w+o\left(1\right).\tag{6.2.42}$$

This proves the result.

□

Arguing as in Proposition 6.1.3, this lemma with $E_N=\{r_N\}$ proves that for any $w\geq 0$ and $\eta<0$,

$$\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log Q^{(N)}_2\left(B^\delta_w\right)\geq\left(\alpha+\eta\right)w.\tag{6.2.43}$$

Since the left hand side is η-independent, we take $\eta\uparrow 0$ to get our result for $\min A_N\ll|\Lambda_N|$. The corresponding result for the $Q^{(N)}_3$-factor follows from taking $E_N=A_N\setminus\{r_N\}$.

For the $\min A_N\gg|\Lambda_N|$ case, the large deviations of the $M^{(3,N)}_1$ component follows in precisely the same way. For $M^{(3,N)}_2$ and $M^{(3,N)}_3$, we take a more direct "hands-on" approach. Let us demonstrate it with $M^{(3,N)}_3$. Since the second-smallest permitted value of $M^{(3,N)}_3$ diverges as $N\to\infty$,

$$Q^{(3)}_{N,\alpha}\left(M^{(3,N)}_3=0\right)=\exp\left(-|\Lambda_N|\sum_{k\in A_N\setminus\{r_N\}}q^{(N)}_k\mathrm{e}^{\alpha k}\right),\tag{6.2.44}$$
$$Q^{(3)}_{N,\alpha}\left(M^{(3,N)}_3>a\right)=1-\exp\left(-|\Lambda_N|\sum_{k\in A_N\setminus\{r_N\}}q^{(N)}_k\mathrm{e}^{\alpha k}\right),\qquad\forall a>0,\tag{6.2.45}$$

for all N sufficiently large. Because $\min A_N \to \infty$ and *(A1)* holds,

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log Q^{(3)}_{N,\alpha} \left(M_3^{(3,N)} = 0\right) = 0. \tag{6.2.46}$$

Now denote $m_N = \min A_N \setminus \{r_N\}$ and note that

$$|\Lambda_N| \sum_{k\in A_N\setminus\{r_N\}} q_k^{(N)} \mathrm{e}^{\alpha k} \leq |\Lambda_N| \mathrm{e}^{\frac{\alpha}{2} m_N} \sum_{k\in A_N\setminus\{r_N\}} q_k^{(N)} \mathrm{e}^{\frac{\alpha}{2} k} \to 0 \tag{6.2.47}$$

as $N \to \infty$. Therefore, for any $a > 0$,

$$\frac{1}{|\Lambda_N|} \log Q^{(3)}_{N,\alpha} \left(M_3^{(3,N)} > a\right) = \frac{1}{|\Lambda_N|} \log \sum_{k\in A_N\setminus\{r_N\}} q_k^{(N)} \mathrm{e}^{\alpha k} + O\left(\frac{1}{|\Lambda_N|} \log|\Lambda_N|\right) \tag{6.2.48}$$

$$\leq \frac{\alpha}{2} \frac{m_N}{|\Lambda_N|} + O\left(\frac{1}{|\Lambda_N|} \log|\Lambda_N|\right) \tag{6.2.49}$$

$$\to -\infty, \tag{6.2.50}$$

as $N \to \infty$. We used $\sum_{k\in A_N\setminus\{r_N\}} q_k^{(N)} \mathrm{e}^{\frac{\alpha}{2} k} < 1$ eventually to show the inequality.

The results (6.2.46) and (6.2.50) can be used to show the large deviation bounds for $M_3^{(3,N)}$ obey the rate function I_3, where

$$I_3(z) = \begin{cases} 0 & : z = 0 \\ +\infty & : z \neq 0. \end{cases} \tag{6.2.51}$$

as required. The $M_2^{(3,N)}$ factor follows similarly with $\{r_N\}$ taking the place of $A_N \setminus \{r_N\}$.

$\square$

We can now bound the PCH energy above and below. Define

$$
\begin{aligned}
&\underline{H} : \mathbb{R}_+^3 \to \mathbb{R} \\
&(x,y,z) \mapsto -(\mu-\alpha)(x+y+z) + \frac{a}{2}(x+y+z)^2 - \frac{b}{2}\left(y^2+z^2\right) && (6.2.52)\\
&\overline{H} : \mathbb{R}_+^3 \to \mathbb{R} \\
&(x,y,z) \mapsto -(\mu-\alpha)(x+y+z) + \frac{a}{2}(x+y+z)^2 - \frac{b}{2}y^2. && (6.2.53)
\end{aligned}
$$

So we have the bounds

$$
|\Lambda_N| \underline{H} \circ \pi_N^{(3)}(\omega) \leq H_N^{(\text{PCH})}(\omega) \leq |\Lambda_N| \overline{H} \circ \pi_N^{(3)}(\omega). \tag{6.2.54}
$$

Significantly, both $\underline{H}$ and $\overline{H}$ are $\mathbb{R}^3$-continuous and bounded below (since $a > b$). Therefore we are free to apply Varadhan's Lemma with each.

Lemma 6.2.9. *Given a closed set $C \subset \mathbb{R}_+^3$,*

$$
\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \int_{\left(\pi_N^{(3)}\right)^{-1}(C)} \mathrm{e}^{-\beta H_N^{(\text{PCH})}} \mathsf{Q}_\alpha(\mathrm{d}\omega) \leq -\inf_C \left\{\mathcal{J}_\alpha + \beta \underline{H}\right\}. \tag{6.2.55}
$$

In particular, the thermodynamic pressure is bounded above as follows:

- *If* $\min A_N \ll |\Lambda_N|$, *and (A2) and (A3) hold,*

$$
p^{(\text{PCH})}(\beta,\mu) \leq \sup_{x,y,z\geq 0} \left\{\mu(x+y+z) - \frac{a}{2}(x+y+z)^2 + \frac{b}{2}\left(y^2+z^2\right) - f(\beta,x)\right\}, \tag{6.2.56}
$$

- *If* $\min A_N \gg |\Lambda_N|$,

$$
p^{(\text{PCH})}(\beta,\mu) \leq \sup_{x\geq 0}\left\{\mu x - \frac{a}{2}x^2 - f(\beta,x)\right\} = p^{(\text{PM})}(\beta,\mu). \tag{6.2.57}
$$

Proof. First we use (6.2.54) and have

$$
\begin{aligned}
\int_{\left(\pi_N^{(3)}\right)^{-1}(C)} \mathrm{e}^{-H_N^{(\text{PCH})}} \mathsf{Q}_\alpha(\mathrm{d}\omega) &\leq \int_{\left(\pi_N^{(3)}\right)^{-1}(C)} \mathrm{e}^{-|\Lambda_N|\underline{H}\circ\pi_N^{(3)}} \mathsf{Q}_\alpha(\mathrm{d}\omega) && (6.2.58)\\
&= \int_C \mathrm{e}^{-|\Lambda_N|\underline{H}} Q_{N,\alpha}^{(3)}(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z). && (6.2.59)
\end{aligned}
$$

Because $\underline{H}$ is a continuous function that is bounded below, this is simply a matter of applying Varadhan's Lemma. Taking $C = \mathbb{R}_+^3$ gives the bound on the thermodynamic pressure.

$\square$

Lemma 6.2.10. *Given an open set $O \subset \mathbb{R}_+^3$,*

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \int_{\left(\pi_N^{(3)}\right)^{-1}(O)} \mathrm{e}^{-\beta H_N^{(PCH)}} \mathsf{Q}_\alpha (\mathrm{d}\omega) \geq - \inf_O \left\{ \mathcal{J}_\alpha + \beta \overline{H} \right\}. \tag{6.2.60}$$

In particular, the thermodynamic pressure is bounded below as follows:

- *If* $\min A_N \ll |\Lambda_N|$, *and (A2) and (A3) hold,*

$$p^{(PCH)}(\beta,\mu) \geq \sup_{x,y,z\geq 0} \left\{ \mu(x+y+z) - \frac{a}{2}(x+y+z)^2 + \frac{b}{2}y^2 - f(\beta,x) \right\}, \tag{6.2.61}$$

- *If* $\min A_N \gg |\Lambda_N|$,

$$p^{(PCH)}(\beta,\mu) \geq \sup_{x\geq 0} \left\{ \mu x - \frac{a}{2}x^2 - f(\beta,x) \right\} = p^{(PM)}(\beta,\mu). \tag{6.2.62}$$

Proof. First we use (6.2.54) and have

$$\int_{\left(\pi_N^{(3)}\right)^{-1}(O)} \mathrm{e}^{-H_N^{(PCH)}} \mathsf{Q}_\alpha (\mathrm{d}\omega) \leq \int_{\left(\pi_N^{(3)}\right)^{-1}(O)} \mathrm{e}^{-|\Lambda_N| \overline{H}\circ \pi_N^{(3)}} \mathsf{Q}_\alpha (\mathrm{d}\omega) \tag{6.2.63}$$

$$= \int_O \mathrm{e}^{-|\Lambda_N|\overline{H}} Q_{N,\alpha}^{(3)} (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z). \tag{6.2.64}$$

Because $\overline{H}$ is a continuous function, this is simply a matter of applying Varadhan's Lemma. Taking $O = \mathbb{R}_+^3$ gives the bound on the thermodynamic pressure.

$\square$

Proof of Theorem 6.2.2. Let us first consider the $\min A_N \ll |\Lambda_N|$ case. The central idea to to apply the Contraction Principle to each of Lemma 6.2.9 and Lemma 6.2.10, and find that the corresponding rate functions coincide.

First note that the pressure bounds from these two lemmas coincide. In both cases the supremum is achieved along $z = 0$ (possibly non-uniquely in the case of

Lemma 6.2.9). To see this, condition on the value of $y+z$ and find the optimal choice(s) of y and z. This proves the limit of the thermodynamic pressure.

From Lemmas 6.2.9 and 6.2.10 we have the following rate functions for the large deviation upper and lower bounds on $Q_{N,\mu}^{(3,PCH)}$:

$$\mathcal{J}^{(U)}(x,y,z) = f(x) - \mu(x+y+z) + \frac{a}{2}(x+y+z)^2 - \frac{b}{2}\left(y^2+z^2\right) + p^{(PCH)}(\mu), \tag{6.2.65}$$

$$\mathcal{J}^{(L)}(x,y,z) = f(x) - \mu(x+y+z) + \frac{a}{2}(x+y+z)^2 - \frac{b}{2}y^2 + p^{(PCH)}(\mu). \tag{6.2.66}$$

To relate the 3-splittings to the 2-splittings, we have the projection $\hat{\pi}:\mathbb{R}_+^3 \to \mathbb{R}_+^2$, $(x,y,z)\mapsto(x,y+z)$. In particular, note that $\hat{\pi}$ is continuous and $\pi_N^{(2)} = \hat{\pi}\circ\pi_N^{(3)}$.

Now as per the Contraction Principle, for $I=U,L$ we define

$$\widehat{\mathcal{J}}^{(I)}(x,y) := \inf_{Y,Z:Y+Z=y} \mathcal{J}^{(I)}(x,Y,Z). \tag{6.2.67}$$

This definition implies that for $I=U,L$ and any $E\subset\mathbb{R}_+^2$,

$$\inf_{(x,y)\in E} \widehat{\mathcal{J}}^{(I)}(x,y) = \inf_{(x,y,z)\in\hat{\pi}^{-1}(E)} \mathcal{J}^{(I)}(x,y,z). \tag{6.2.68}$$

Since $\hat{\pi}$ is continuous, the set $\hat{\pi}^{-1}(E)$ is an open (closed) subset of $\mathbb{R}_+^3$ for any open (closed) $E\subset\mathbb{R}_+^2$. Therefore the large deviation upper and lower bounds for $Q_{N,\mu}^{(3,PCH)}\circ\hat{\pi}^{-1}$ follow from Lemmas 6.2.9 and 6.2.10, and (6.2.68). However

$$Q_{N,\mu}^{(3,PCH)}\circ\hat{\pi}^{-1} = \mathsf{Q}_{N,\mu}^{(PCH)}\circ\left(\pi_N^{(3)}\right)^{-1}\circ\hat{\pi}^{-1} = \mathsf{Q}_{N,\mu}^{(PCH)}\circ\left(\hat{\pi}\circ\pi_N^{(3)}\right)^{-1} = Q_{N,\mu}^{(2,PCH)}, \tag{6.2.69}$$

and so we in fact have large deviation upper and lower bounds for $Q_{N,\mu}^{(2,PCH)}$.

Finally, by explicitly calculating $\widehat{\mathcal{J}}^{(U)}$ and $\widehat{\mathcal{J}}^{(L)}$, we find that

$$\widehat{\mathcal{J}}^{(U)}(x,y) = \widehat{\mathcal{J}}^{(L)}(x,y) = \mathcal{J}^{(PCH)}(x,y), \tag{6.2.70}$$

and we therefore have a full LDP.

The $\min A_N \gg |\Lambda_N|$ case follows similarly with the corresponding rate functions $\mathcal{J}^{(U)}$ and $\mathcal{J}^{(L)}$ for that case.

$\square$

Remark 6.2.11. *This expression for the pressure also illustrates where the lower semicontinuous regularisation of the energy comes from in [AD18]. Here we are free to choose y to minimise the energy because the entropy (via the free energy) does not see this extra particle mass.* $\diamond$

Remark 6.2.12. *This expression for the PCH pressure is identical to the expression derived by Lewis for the true (momentum-space) HYL model if their integrated density of states satisfies $\lambda_0 = 0$.* $\diamond$

6.3 Momentum-Space HYL Interaction

An approach similar to the above can be applied to replicate the work of [BLP88] in deriving the thermodynamic pressure of their model. The main challenge is in calculating the logarithmic moment generating function for the appropriate mass splitting in the ideal gas model. This uses techniques like those used in [BLP82] for proving the convergence of the thermodynamic pressure of the free Bose gas.

In [BLP88], they begin by trying to derive a LDP for the free Bose gas with the ground state split off from the rest of the gas. The lower bound for their HYL model then followed by applying Varadhan's lemma with an appropriate tilt (very similar to the one used above). The upper bound was more involved. The higher states were shifted down and the resulting error was bounded using a combinatorial argument - it was here that a bound on the number of states that could be affected by the counter-term in the HYL model arose. Finally an application of Varadhan's lemma with an appropriate tilt gave the bound. Our first improvement on their work is a minor one. In our notation, they only consider $a = 2b$. They do this because that is the physical case for the true HYL interaction energy, but their argument works perfectly well for more general $a > b > 0$. However, this simplification hides the different roles that the mean field and the counter terms play. By considering the more general scenario we will be able to get greater insight into their respective roles. Our more significant improvement is in circumventing the crude combinatorial aspect of their proof and arguing purely in terms of large deviation techniques. This allows us to consider more general mass splittings than [BLP88] and therefore we can consider partial HYL models that do not just affect *consecutive* low energy states. This line of arguing also highlights why [BLP88] has the bound on how quickly the number affected states must grow - the free energy of these states needs to vanish.

First let us describe the probabilistic setting. Let Ω be the space of terminating sequences of non-negative integers: $\omega \in \Omega$ is a sequence $\{\omega_j \in \mathbb{N} : j = 1, 2, \ldots\}$. The basic random variables are the evaluation maps $\sigma_j : \Omega \to \mathbb{N}$ given by $\sigma_j(\omega) = \omega_j$. These are the occupation numbers of each state.

We then consider a sequence of measures on Ω, indexed by $N \in \mathbb{N}$. These require some parameters with associated conditions. For each N, let $\left\{\lambda_j^{(N)}\right\}_{j\in\mathbb{N}}$ be a non-decreasing sequence with $\lambda_1^{(N)} = 0$, associated with the finite volume region $\Lambda_N \subset \mathbb{R}^d$.

Remark 6.3.1. *In the physical models, these $\lambda_k^{(N)}$ represent the energy differences of the eigenvalues of the single-particle Hamiltonians from the ground state. The equivalence of the grand canonical pressure and mass distributions for the free Bose gas is argued in [BLP82], and follows for other models similarly. Note that the boundary conditions of the model are encoded in these $\lambda_k^{(N)}$, so the following discussion will apply for any boundary conditions such that the corresponding $\lambda_k^{(N)}$ satisfy the required conditions.* $\diamond$

Since Ω is a countable set, we can define the grand canonical free Bose gas measures on Ω for $\alpha < 0$ with

$$\mathbb{P}_{N,\alpha}(\omega) = \frac{1}{Z_N(\beta,\alpha)} \exp\left(\beta \sum_{j\in\mathbb{N}} \left(\alpha - \lambda_j^{(N)}\right) \sigma_j(\omega)\right), \tag{6.3.1}$$

where

$$Z_N(\beta,\alpha) = \sum_{\omega\in\Omega} \exp\left(\beta \sum_{j\in\mathbb{N}} \left(\alpha - \lambda_j^{(N)}\right) \sigma_j(\omega)\right) = \prod_{j\in\mathbb{N}} \frac{1}{1 - \mathrm{e}^{\beta\left(\alpha - \lambda_j^{(N)}\right)}}. \tag{6.3.2}$$

For chemical potential $\alpha < 0$, the finite volume free gas pressure $p_N(\beta,\alpha)$

can be written as

$$p_N(\beta,\alpha) = \frac{1}{\beta|\Lambda_N|}\log Z_N(\beta,\alpha) = -\frac{1}{\beta|\Lambda_N|}\sum_{j\in\mathbb{N}}\log\left(1-\mathrm{e}^{\beta\left(\alpha-\lambda_j^{(N)}\right)}\right). \qquad (6.3.3)$$

For the limit $p(\beta,\alpha) = \lim_{N\to\infty} p_N(\beta,\alpha)$ to exist, we require the conditions

(P1): $\varphi(\beta) = \lim_{N\to\infty}\frac{1}{|\Lambda_N|}\sum_{j\in\mathbb{N}}\mathrm{e}^{-\beta\lambda_j^{(N)}}$ exists for all $\beta\in(0,+\infty)$,

(P2): $\exists\beta'\in(0,+\infty)$ such that $\varphi(\beta')\neq 0$.

If we define the partial pressure

$$p_\beta(\alpha|\lambda) = -\log\left(1-\mathrm{e}^{\beta(\alpha-\lambda)}\right) \qquad (6.3.4)$$

and the distribution function

$$F^{(N)}(\lambda) = \frac{1}{|\Lambda_N|}\#\left\{j\in\mathbb{N} : \lambda_j^{(N)}\leq\lambda\right\}, \qquad (6.3.5)$$

then the finite volume pressures can be written as

$$p_N(\beta,\alpha) = \int_{[0,\infty)} p_\beta(\alpha|\lambda)\,\mathrm{d}F^{(N)}(\lambda). \qquad (6.3.6)$$

From the Laplace transform techniques of [Fel68], when *(P1)* holds, there exists a unique distribution function F such that

$$\varphi(\beta) = \int_{[0,\infty)}\mathrm{e}^{-\beta\lambda}\mathrm{d}F(\lambda) \qquad (6.3.7)$$

and $F^{(N)}(\lambda)\to F(\lambda)$, at least at the points of continuity of F. We call F the *integrated density of states*. The following result from [BLP82] gives the existence of $p(\beta,\alpha)$.

Proposition 6.3.2. *Suppose that (P1) and (P2) hold. Then the limit $p(\beta,\alpha)$ exists, and is given by*

$$p(\beta,\alpha) = \begin{cases}\int_{[0,\infty)} p_\beta(\alpha|\lambda)\,\mathrm{d}F(\lambda) & :\alpha<0\\ +\infty & :\alpha\geq 0.\end{cases} \qquad (6.3.8)$$

Proof. This is given in [BLP82, Theorem 1].

□

Remark 6.3.3. *The divergence of $p_N(\beta,\alpha)$ for $\alpha \geq 0$ is clear from $\lambda_1^{(N)} \equiv 0$: $p_N(\beta,\alpha) = +\infty$ for $\alpha \geq 0$. This is despite the possibility that $p_\beta(\alpha|\cdot)$ could be integrable with respect to F on $[0,\infty)$.* ◇

The HYL measures are then found by taking the non-interacting free Bose measures and applying a tilt via an interaction Hamiltonian.

Definition 6.3.4. For $a > b > 0$, and non-empty $A_N \subset \mathbb{N}$, we define the *partial momentum HYL* (or PMH) energy as

$$H_N^{(PMH)}(\omega) = -(\mu-\alpha)\sum_{j\in\mathbb{N}}\sigma_j(\omega) + \frac{a}{2|\Lambda_N|}\left(\sum_{j\in\mathbb{N}}\sigma_j(\omega)\right)^2 - \frac{b}{2|\Lambda_N|}\sum_{j\in A_N}(\sigma_j(\omega))^2. \tag{6.3.9}$$

Then

$$\mathbb{P}_N^{(PMH)}(\omega) = \frac{\exp\left(-\beta H_N^{(PMH)}(\omega)\right)}{\mathbb{E}_{N,\alpha}\left[\exp\left(-\beta H_N^{(PMH)}\right)\right]}\mathbb{P}_{N,\alpha}(\omega). \tag{6.3.10}$$

Like for the cycle-space version, $\mathbb{P}_N^{(PMH)}$ is independent of α, so we suppress it from the relevant notation.

We aim to find an expression for the thermodynamic pressure:

$$p^{(PMH)}(\beta,\mu) = p(\beta,\alpha) + \lim_{N\to\infty}\frac{1}{\beta|\Lambda_N|}\log\mathbb{E}_{N,\alpha}\left[\mathrm{e}^{-\beta H_N^{(PMH)}}\right]. \tag{6.3.11}$$

In addition to the macro-scale variables previously mentioned, our expression will also depend upon the micro-scale values

$$\lambda_0 := \liminf_{N\to\infty}\lambda^{(N)}_{\min\mathbb{N}\setminus A_N}, \qquad \lambda_+ := \liminf_{N\to\infty}\lambda^{(N)}_{\min A_N}. \tag{6.3.12}$$

To prove the existence of $p^{(PMH)}$, we require the conditions

(H1): $\varphi^+(\beta) = \lim_{N\to\infty}\frac{1}{|\Lambda_N|}\sum_{j\in A_N}\mathrm{e}^{-\beta\lambda_j^{(N)}} = 0$ for all $\beta\in(0,+\infty)$,

(H2): λ_0 is a point of continuity of F,

(H3): there exists a sequence $\{m_N\}_{N\in\mathbb{N}}$ such that $m_N \in A_N$ and

$$\liminf_{N\to\infty} \lambda^{(N)}_{\min \mathbb{N}\setminus\{m_N\}} = \lambda_0 \qquad \text{and} \qquad \liminf_{N\to\infty} \lambda^{(N)}_{m_N} = \lambda_+, \tag{6.3.13}$$

in addition to *(P1)* and *(P2)*.

Remark 6.3.5. *We shall find it useful to have the following expression:*

$$f(\beta, x; \lambda_0) = \sup_{s<0} \left\{ sx - \int_{[0,\infty)} p_\beta (s + \lambda_0 | \lambda) \,\mathrm{d}F(\lambda) \right\} \tag{6.3.14}$$

$$= \sup_{s<\lambda_0} \left\{ sx - \int_{[\lambda_0,\infty)} p_\beta (s | \lambda) \,\mathrm{d}F(\lambda) \right\} - \lambda_0 x. \tag{6.3.15}$$

This last equality holds from shifting the supremum index and the supremum in the second expression is equal to the conditional free energy in [Lew86]. For $\lambda_0 = 0$, it is equal to the thermodynamic free energy of the ideal Bose gas (also see [Lew86]). $\diamond$

Upper bound. We first aim to prove an upper bound on the thermodynamic pressure by considering a splitting of the mass by cycle lengths and an energy minorant that is diagonal in this mass partition.

Let $\left(M_0^{(N)}, M_+^{(N)}\right)$ be random variables, where

$$M_0^{(N)} = \frac{1}{|\Lambda_N|} \sum_{j\in\mathbb{N}\setminus A_N} \sigma_j, \qquad M_+^{(N)} = \frac{1}{|\Lambda_N|} \sum_{j\in A_N} \sigma_j, \tag{6.3.16}$$

so $M_+^{(N)}$ is the particle density supported on A_N-eigenstates, and $M_0^{(N)}$ is the remaining density supported on finite momentum eigenstates. Let $Q^{A_N}_{N,\alpha}$ be the measure of $\left(M_0^{(N)}, M_+^{(N)}\right)$ on $\mathbb{R}_+^2$ induced by the independent geometric distributions of the σ_k.

Lemma 6.3.6. *Suppose (P1), (P2), (H1) and (H2) hold. Then for $\alpha < 0$, the* *sequence of measures* $\left\{Q_{N,\alpha}^{A_N}\right\}_{N\in\mathbb{N}}$ *has the logarithmic moment generating function*

$$\mathcal{L}\left(s,t\right)\begin{cases}=\beta\int_{[0,\infty)}p_\beta\left(\alpha+\frac{s}{\beta}|\lambda\right)\mathrm{d}F\left(\lambda\right)-\beta p\left(\beta,\alpha\right) & :\frac{s}{\beta}<\alpha+\lambda_0 \textit{ and } \frac{t}{\beta}<-\alpha+\lambda_+,\\ =+\infty & :\frac{s}{\beta}>-\alpha+\lambda_0 \textit{ or } \frac{t}{\beta}>-\alpha+\lambda_+,\\ \geq\beta\int_{[0,\infty)}p_\beta\left(\alpha+\frac{s}{\beta}|\lambda\right)\mathrm{d}F\left(\lambda\right)-\beta p\left(\beta,\alpha\right) & : \textit{otherwise},\end{cases} \tag{6.3.17}$$

given as a $\limsup$ *for "otherwise".*

The Legendre-Fenchel transform is given by

$$\mathcal{L}^*\left(x,y\right)=\beta f\left(\beta,x;\lambda_0\right)-\beta\left(\alpha-\lambda_0\right)x-\beta\left(\alpha-\lambda_+\right)y+\beta p\left(\beta,\alpha\right). \tag{6.3.18}$$

Proof. From $M_0^{(N)}$ and $M_+^{(N)}$ being sums of independent geometric random variables, we have

$$\mathcal{L}\left(s,t\right):=\limsup_{N\to\infty}\frac{1}{|\Lambda_N|}\log Q_{N,\alpha}^{A_N}\left[\mathrm{e}^{|\Lambda_N|\left(sM_0^{(N)}+tM_+^{(N)}\right)}\right] \tag{6.3.19}$$

$$=\limsup_{N\to\infty}\frac{1}{|\Lambda_N|}\left[\log\mathbb{E}_{N,\alpha}\left[\mathrm{e}^{|\Lambda_N|sM_0^{(N)}}\right]+\log\mathbb{E}_{N,\alpha}\left[\mathrm{e}^{|\Lambda_N|tM_+^{(N)}}\right]\right] \tag{6.3.20}$$

$$=\limsup_{N\to\infty}\frac{1}{|\Lambda_N|}\left[\sum_{j\in\mathbb{N}}\log\left(1-\mathrm{e}^{\alpha-\lambda_j^{(N)}}\right)-\sum_{j\in\mathbb{N}\setminus A_N}\log\left(1-\mathrm{e}^{s+\alpha-\lambda_j^{(N)}}\right)\right.$$
$$\left.-\sum_{j\in A_N}\log\left(1-\mathrm{e}^{t+\alpha-\lambda_j^{(N)}}\right)\right]. \tag{6.3.21}$$

If $s+\alpha>\lambda_0$ or $t+\alpha>\lambda_+$ then the first terms in the respective sums equal $+\infty$ for N sufficiently large.

To evaluate the limits for $s+\alpha<\lambda_0$ and $t+\alpha>\lambda_+$, we follow the approach of [BLP82] and use the Laplace transform techniques of [Fel68]. In addition to the distribution function $F^{(N)}$ defined in (6.3.5), we denote

$$F_0^{(N)}\left(\lambda\right)=\frac{1}{|\Lambda_N|}\#\left\{j\in\mathbb{N}\setminus A_N:\lambda_j^{(N)}\leq\lambda\right\}, \tag{6.3.22}$$

$$F_+^{(N)}\left(\lambda\right)=\frac{1}{|\Lambda_N|}\#\left\{j\in A_N:\lambda_j^{(N)}\leq\lambda\right\}. \tag{6.3.23}$$

Then

$$\mathcal{L}(s,t) = \limsup_{N\to\infty} \left[\int_{[0,+\infty)} p(s+\alpha|\lambda)\, \mathrm{d}F_0^{(N)}(\lambda) + \int_{[0,+\infty)} p(t+\alpha|\lambda)\, \mathrm{d}F_+^{(N)}(\lambda) - \int_{[0,+\infty)} p(\alpha|\lambda)\, \mathrm{d}F^{(N)}(\lambda) \right]. \quad (6.3.24)$$

Note that because of *(P1)* and *(H1)*, we know that the limiting Laplace transform of $F_0^{(N)}$ equals $\varphi(\beta)$ and therefore - except perhaps at points of discontinuity of F - we have $F_0^{(N)}(\lambda) \to F(\lambda)$.

Let $u < \lambda_0$. Because $p(u|\lambda)$ is continuous and bounded for $\lambda \in [\lambda_0, \infty)$, and λ_0 is a point of continuity of F,

$$\lim_{N\to\infty} \int_{[\lambda_0,+\infty)} p(u|\lambda)\, \mathrm{d}F_0^{(N)}(\lambda) = \int_{[\lambda_0,+\infty)} p(u|\lambda)\, \mathrm{d}F(\lambda). \quad (6.3.25)$$

Also $\lambda_0 = \liminf_{N\to\infty} \lambda^{(N)}_{\min \mathbb{N}\setminus A_N} = \liminf_{N\to\infty} \min_{j\in\mathbb{N}\setminus A_N} \lambda_j^{(N)}$ is a point of continuity of F, so

$$F_0^{(N)}(\lambda_0) \to F(\lambda_0) = 0. \quad (6.3.26)$$

For each $u' \in (u, \lambda_0)$, $\min_{j\in\mathbb{N}\setminus A_N} \lambda_j^{(N)} > u'$ eventually, and $p(u|\lambda)$ is bounded above on $\lambda \in (u', \lambda_0)$ by $p(u|u') < +\infty$. Therefore

$$\int_{[0,\lambda_0)} p(u|\lambda)\, \mathrm{d}F_0^{(N)}(\lambda) = \int_{[u',\lambda_0)} p(u|\lambda)\, \mathrm{d}F_0^{(N)}(\lambda) \le p(u|u')\, F_0^{(N)}(\lambda_0) \to 0 \quad (6.3.27)$$

for all $u < \lambda_0$. This means

$$\lim_{N\to\infty} \int_{[0,+\infty)} p(u|\lambda)\, \mathrm{d}F_0^{(N)}(\lambda) = \int_{[0,+\infty)} p(u|\lambda)\, \mathrm{d}F(\lambda), \qquad \forall u < \lambda_0. \quad (6.3.28)$$

For the integral over $F^{(N)}$, let $u < 0$ so we have

$$\lim_{N\to\infty}\int_{[0,+\infty)} p\left(u|\lambda\right)\mathrm{d}F^{(N)}\left(\lambda\right) = \lim_{N\to\infty}\int_{[-\epsilon,+\infty)} p\left(u|\lambda\right)\mathrm{d}F^{(N)}\left(\lambda\right) \qquad \forall\epsilon\in(0,-u) \tag{6.3.29}$$

$$= \int_{[-\epsilon,+\infty)} p\left(u|\lambda\right)\mathrm{d}F\left(\lambda\right) \tag{6.3.30}$$

$$= \int_{[0,+\infty)} p\left(u|\lambda\right)\mathrm{d}F\left(\lambda\right). \tag{6.3.31}$$

This follows because $F^{(N)}\left(\lambda\right) = 0$ and $F\left(\lambda\right) = 0$ for all $\lambda < 0$. Therefore $-\epsilon$ is a point of continuity of F.

For the integral over $F_+^{(N)}$, we will use the Laplace transform of $F_+^{(N)}$ as an upper bound. Let $u < \lambda_+$ and $u' \in (u, \lambda_+)$, so

$$\lim_{N\to\infty}\int_{[0,+\infty)} p\left(u|\lambda\right)\mathrm{d}F_+^{(N)}\left(\lambda\right) = \lim_{N\to\infty}\int_{[u',+\infty)} p\left(u|\lambda\right)\mathrm{d}F_+^{(N)}\left(\lambda\right). \tag{6.3.32}$$

Now by differentiating we know that $p\left(u|\lambda\right) \leq p\left(u|u'\right)\mathrm{e}^{u'-\lambda}$ for all $u < u' \leq \lambda$. Hence

$$0 \leq \lim_{N\to\infty}\int_{[u',+\infty)} p\left(u|\lambda\right)\mathrm{d}F_+^{(N)}\left(\lambda\right) \leq p\left(u|u'\right)\mathrm{e}^{u'}\varphi_+\left(1\right) = 0, \tag{6.3.33}$$

by *(H1)*.

As for $u = \lambda_0$ or $u = \lambda_+$, we note that the finite N logarithmic moment generating functions are all non-decreasing in s and in t. Therefore we know that the limit superior is also non-decreasing in s and in t. This gives the required result for $\mathcal{L}\left(s,t\right)$.

Now

$$\mathcal{L}^*(x,y) = \sup_{s,t\in\mathbb{R}} \{sx + ty - \mathcal{L}(s,t)\} \tag{6.3.34}$$

$$= \sup_{s<-\alpha+\lambda_0} \left\{ sx - \int_{[0,\infty)} p(\alpha+s|\lambda)\,\mathrm{d}F(\lambda) \right\} + \sup_{t<-\alpha+\lambda_+} \{ty\} + p(\alpha) \tag{6.3.35}$$

$$= \sup_{s<-\alpha+\lambda_0} \left\{ sx - \int_{[0,\infty)} p(\alpha+s|\lambda)\,\mathrm{d}F(\lambda) \right\} - (\alpha-\lambda_+)\,y + p(\alpha)\,. \tag{6.3.36}$$

Adding and subtracting $(\alpha-\lambda_0)\,x$ and shifting the supremum index gives the required expression.

□

Lemma 6.3.7. *Suppose (P1), (P2), (H1), and (H2) hold. Then for $\alpha < 0$, the sequence of measures $\left\{Q_{N,\alpha}^{A_N}\right\}$ obeys a LDP with rate $|\Lambda_N|$ and good rate function*

$$\mathcal{J}_\alpha(x,y) = \beta f(\beta,x;\lambda_0) - \beta(\alpha-\lambda_0)\,x - \beta(\alpha-\lambda_+)\,y + \beta p(\beta,\alpha)\,. \tag{6.3.37}$$

Proof. This proof proceeds similarly to that of Lemma 6.2.7. The large deviation upper bound holds because $0 \in \operatorname{int}(\mathcal{D}_{\mathcal{L}})$ gives exponential tightness, and we split $Q_{N,\alpha}^{A_N}$ into its marginals $Q_{N,\alpha}^{0}$ and $Q_{N,\alpha}^{+}$ and prove the large deviation lower bounds for each independently. For the strictly convex parts of $\mathcal{L}_0^*(x) = f(x;\lambda_0) - (\alpha-\lambda_0) + p(\alpha)$ (that is, $x < \varrho_c$), the standard techniques suffice. However, we once again need an additional lemma for the linear $\mathcal{L}_+^*(y) = -(\alpha-\lambda_+)\,y$ and the affine parts of $\mathcal{L}_0^*(x)$.

Lemma 6.3.8. *Let N_1 and N_2 be independent non-negative integer valued random variables with means m_1 and m_2 respectively. Suppose that N_1 is geometrically distributed and that $\Delta \geq 1$. Then*

$$\mathbb{P}\left(N_1 + N_2 \in B_{m_1+m_2}^{\Delta}\right) \geq \frac{1}{m_1+m_2}\left(\frac{m_1}{m_1+1}\right)^{m_1+m_2+2}. \tag{6.3.38}$$

Proof of Lemma 6.3.8. This is given in [BLP88, Lemma A1].

$\square$

Our proof from here is slightly different to those used in the cycle-space models. We mimic the proof of [BLP88, Theorem A1]. Given $y \geq \varrho_c$, choose a tilt t_N such that $\mathbb{E}_N\left[\sum_{k\in\mathbb{N}\setminus A_N}\sigma_k\right] = y|\Lambda_N|$. By [BLP88, Proposition 2], $\alpha - \lambda_0 + t_N \to 0$. Like in our discussion of the periodic boundary condition in Proposition 6.1.3, we perturb - if necessary - t_N by an arbitrarily small amount so that $\alpha - \lambda_0 + t_N \neq 0$ and we have $\mathcal{L}$ as a limit. Then we choose $N_1 = \sigma_{\min\mathbb{N}\setminus A_N}$ and $N_2 = \sum_{k\in\mathbb{N}\setminus A_N : k>\min\mathbb{N}\setminus A_N}\sigma_k$, so

$$\frac{m_1}{m_1+1} = \mathrm{e}^{\alpha-\lambda_0+t_N}, \qquad m_1+m_2 = |\Lambda_N|y \tag{6.3.39}$$

and

$$\tilde{Q}^0_{N,\alpha}\left(B^\delta_y\right) \geq \frac{1}{|\Lambda_N|y}\mathrm{e}^{(\alpha-\lambda_0+t_N)(|\Lambda_N|y+2)} \tag{6.3.40}$$

for $|\Lambda_N| \geq 1/\delta$. Therefore

$$\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log\tilde{Q}^0_{N,\alpha}\left(B^\delta_y\right) = 0 \tag{6.3.41}$$

as required. The corresponding result for $Q^+_{N,\alpha}$ follows similarly.

$\square$

Now we bound the HYL energy below as follows:

$$\begin{aligned} H^{(PMH)}_N(\omega) \geq \underline{H_N}(\omega) := |\Lambda_N|\Big[&-(\mu-\alpha)\left(M^{(N)}_0(\omega)+M^{(N)}_+(\omega)\right) \\ &+\frac{a}{2}\left(M^{(N)}_0(\omega)+M^{(N)}_+(\omega)\right)^2 - \frac{b}{2}\left(M^{(N)}_+(\omega)\right)^2\Big]. \end{aligned} \tag{6.3.42}$$

Significantly, this bound is $\mathbb{R}^2$-continuous in $\left(M^{(N)}_0, M^{(N)}_+\right)$ and bounded below on $\mathbb{R}^2_+$ (since $a > b$). Therefore we are free to apply Varadhan's Lemma.

Lemma 6.3.9. *The thermodynamic pressure is bounded above as follows:*

$$p^{(PMH)}(\beta,\mu) \leq \sup_{x,y\geq 0}\left\{\mu(x+y) - \frac{a}{2}(x+y)^2 + \frac{b}{2}y^2 - f(\beta,x;\lambda_0) - \lambda_0 x - \lambda_+ y\right\}. \tag{6.3.43}$$

Proof. Because we have bounded the energy tilt below with a continuous function that is bounded below, this is simply a matter of applying Varadhan's Lemma with tilt $\underline{H_N}$ and the adjusted measure $Q^{A_N}_{N,\alpha}$.

$\square$

Lower Bound. We now prove a lower bound on the thermodynamic pressure by considering a different splitting of the mass by momentum eigenstate with an energy minorant that is diagonal in this mass partition.

For all $N \in \mathbb{N}$ let $m_N \in A_N$, and define

$$\lambda'_0 := \liminf_{N\to\infty} \lambda^{(N)}_{\min \mathbb{N}\setminus\{m_N\}}, \qquad \lambda'_+ := \liminf_{N\to\infty} \lambda^{(N)}_{m_N}. \tag{6.3.44}$$

Let $\left(M'^{(N)}_0, M'^{(N)}_+\right)$ be a random variable, where

$$M'^{(N)}_0 = \frac{1}{|\Lambda_N|} \sum_{j\in\mathbb{N}\setminus\{m_N\}} \sigma_j, \qquad M'^{(N)}_+ = \frac{1}{|\Lambda_N|}\sigma_{m_N}. \tag{6.3.45}$$

Let $Q^{m_N}_{N,\alpha}$ be the measure of $\left(M'^{(N)}_0, M'^{(N)}_+\right)$ on $\mathbb{R}^2_+$ induced by the independent geometric distributions of the σ_k.

Lemma 6.3.10. *Suppose (P1), (P2) hold. For $\alpha < 0$, the sequence of measures $\left\{Q^{m_N}_{N,\alpha}\right\}_{N\in\mathbb{N}}$ obeys a LDP with rate $|\Lambda_N|$ and good rate function*

$$\underline{\mathcal{J}_\alpha}(x, y) = \beta f(\beta, x; \lambda'_0) - \beta(\alpha - \lambda'_0)x - \beta\left(\alpha - \lambda'_+\right)y + \beta p(\beta, \alpha). \tag{6.3.46}$$

Proof. This is nearly a corollary of Lemma 6.3.6 and Lemma 6.3.7. Since $\{m_N\}$ consists of only one state, *(H1)* holds. We don't need *(H2)* though, because $\lambda'_0 = 0$ or $\lambda'_0 = \liminf_{N\to\infty}\lambda^{(N)}_2$. This means that at any point that our region of integration ends at λ_0, we can extend it below to a point of continuity of F without changing the value of the integral (much like in step (6.3.29)).

$\square$

Now we bound the HYL tilt above as follows:

$$\begin{aligned} H^{(PMH)}_N(\omega) \le \overline{H_N}(\omega) := |\Lambda_N| \Big[&-(\mu-\alpha)\left(M'^{(N)}_0(\omega) + M'^{(N)}_+(\omega)\right) \\ &+ \frac{a}{2}\left(M'^{(N)}_0(\omega) + M'^{(N)}_+(\omega)\right)^2 - \frac{b}{2}\left(M'^{(N)}_+(\omega)\right)^2\Big]. \end{aligned} \tag{6.3.47}$$

Significantly, this bound is $\mathbb{R}^2$-continuous in $\left(M_0'^{(N)}, M_+'^{(N)}\right)$ and bounded below on $\mathbb{R}_+^2$ (since $a > b$). Therefore we are free to apply Varadhan's Lemma.

Lemma 6.3.11. *The thermodynamic pressure is bounded below as follows:*

$$p^{(PMH)}(\beta, \mu) \geq \sup_{x,y \geq 0} \left\{ \mu (x+y) - \frac{a}{2}(x+y)^2 + \frac{b}{2} y^2 - f(\beta, x; \lambda_0') - \lambda_0' x - \lambda_+' y \right\}. \tag{6.3.48}$$

Proof. Because we have bounded the energy tilt above with a continuous function that is bounded below, this is simply a matter of applying Varadhan's Lemma.

□

Theorem 6.3.12. *Suppose conditions (P1), (P2), (H1), (H2), and (H3) hold. Then for all $\mu \in \mathbb{R}$ and $a > b > 0$,*

$$p^{(PMH)}(\beta, \mu) = \sup_{x,y \geq 0} \left\{ \mu (x+y) - \frac{a}{2}(x+y)^2 + \frac{b}{2} y^2 - f(\beta, x; \lambda_0) - \lambda_0 x - \lambda_+ y \right\} \tag{6.3.49}$$

$$= \sup_{x \geq 0} \left\{ (\mu - \lambda_+ - \lambda_0) x - \frac{a}{2} x^2 + \frac{(\mu - \lambda_+ - ax)_+^2}{2(a-b)} - f(\beta, x; \lambda_0) \right\}. \tag{6.3.50}$$

Proof. Since the conditions *(P1)*, *(P2)*, *(H1)*, and *(H2)* hold, we are able to apply Lemma 6.3.9 and Lemma 6.3.11. We only need top show that we can choose a sequence of $m_N \in A_N$ such that $\lambda_0 = \lambda_0'$ and $\lambda_+ = \lambda_+'$. This is precisely what *(H3)* says.

□

We are now able to compare the assumptions required by [BLP88] in their derivation of the PMH thermodynamic pressure with the assumptions required by our above derivation. We first formulate our result in comparable terms.

Corollary 6.3.13. *Suppose $A_N = \{1, \ldots, c_N\}$ where $c_n \ll |\Lambda_N|$, and conditions (P1) and (P2) hold. Then if $\lambda_0 = \liminf_{N\to\infty} \lambda_{c_N+1}^{(N)} = \liminf_{N\to\infty} \lambda_2^{(N)}$ and λ_0 is a* *point of continuity of F, then*

$$p^{(PMH)}(\beta,\mu)=\sup_{x,y\geq 0}\left\{\mu(x+y)-\frac{a}{2}(x+y)^2+\frac{b}{2}y^2-f(\beta,x;\lambda_0)-\lambda_0 x\right\}. \quad (6.3.51)$$

Proof. Condition *(H1)* immediately follows from having $c_N \ll |\Lambda_N|$. Condition *(H2)* follows from $\lambda^{(N)}_{\min \mathbb{N}\setminus A_N}=\lambda^{(N)}_{c_N+1}$. Condition *(H3)* follows from having $\liminf_{N\to\infty}\lambda^{(N)}_{c_N+1}=\liminf_{N\to\infty}\lambda^{(N)}_2$ and choosing $m_N\equiv 1$. Finally note that with this A_N we necessarily have $\lambda_+=0$.

□

Corollary 6.3.14. *Suppose $A_N=\{1,\ldots,c_N\}$ where $c_n \ll |\Lambda_N|$, and conditions (P1) and (P2) hold. Then if $\lambda_0=\lim_{N\to\infty}\lambda^{(N)}_2$ exists and is a point of continuity of F, then*

$$p^{(PMH)}(\beta,\mu)=\sup_{x,y\geq 0}\left\{\mu(x+y)-\frac{a}{2}(x+y)^2+\frac{b}{2}y^2-f(\beta,x;\lambda_0)-\lambda_0 x\right\}. \quad (6.3.52)$$

Proof (Sketch). The techniques used in proving Theorem 6.3.12 allow us to complete the proof of [BLP88]. The lower bound is proven in the same way as above, with the splitting $\left(M'^{(N)}_0, M'^{(N)}_+\right)$ and the tilt $\overline{H_N}$. The upper bound in [BLP88] has the extra aspect of the combinatorial bound on the size of the set $\left\{\{n_1,\ldots,n_{c_N}\}\in\mathbb{N}^{c_N}:\sum_{j=1}^{c_N}n_j=m\right\}$ which - with a shifting of the states by c_N positions - enables us to approximate the bottom c_N states with a single state and leave the integrated density of states unchanged. The required LDP aspect of this argument follows from using Lemma 6.3.7 with $A_N=\{1\}$. This is why we only need the condition on $\lim_{N\to\infty}\lambda^{(N)}_2$, and not $\liminf_{N\to\infty}\lambda^{(N)}_{c_N+1}$ as in Corollary 6.3.13.

□

Remark 6.3.15. *In [BLP88], they remark that they would like to derive the pressure of the full momentum HYL (FMH) model. This is the same as the PMH model with the interaction energy $H^{(PMH)}_N$ replaced with $H^{(FMH)}_N$, where*

$$H^{(FMH)}_N(\omega)=-(\mu-\alpha)\sum_{j\in\mathbb{N}}\sigma_j(\omega)+\frac{a}{2|\Lambda_N|}\left(\sum_{j\in\mathbb{N}}\sigma_j(\omega)\right)^2-\frac{b}{2|\Lambda_N|}\sum_{j\in\mathbb{N}}(\sigma_j(\omega))^2. \quad (6.3.53)$$

At that point in time they were unable to overcome the technical difficulties arising from the counter-term affecting all energies, but they succeeded in [BDLP90b]. The PMH model pressure naturally provides a lower bound for the FMH model pressure, but they were able to get the upper bound by changing the topology on which their LDP lived. In the ℓ_1 topology considered above, the configuration with all the particles are in the ground state and the state with all the particles in the second lowest state are far apart. In [BDLP90b], they consider the empirical density measure on $\mathcal{M}_+^b(\mathbb{R}_+)$, the space of bounded positive measures, equipped with the narrow topology, that is the weakest topology such that the mapping

$$m \mapsto \langle m, f \rangle = \int_{[0,\infty)} f(\lambda)\, m(\mathrm{d}\lambda) \tag{6.3.54}$$

is continuous for every bounded, continuous f. In particular, this ensures that the two configurations described above are close if the momentum eigenvalues corresponding to the two states are close. This changes the continuity properties of the HYL counter-term and the result follows from Varadhan's Lemma. As we show in Lemma 6.3.16, this equality is not the case for our partial and full cycle HYL models. The argument of Lemma 6.3.16 fails for the momentum space model because the 'high energy' states are not macroscopically occupied in the limit, like our short cycles are. Therefore the partial model expectation of the extra energy vanishes in the limit. ◇

Lemma 6.3.16. *For all $\mu \in \mathbb{R}$ and $\beta > 0$, the full and partial cycle-space HYL pressures differ:*

$$p^{(\text{FCH})}(\beta, \mu) > p^{(\text{PCH})}(\beta, \mu)\,. \tag{6.3.55}$$

PROOF. By rearranging terms and using Jensen's inequality,

$$\frac{1}{|\Lambda_N|}\log \mathbb{E}_N\left[\mathrm{e}^{-|\Lambda_N|H^{(FCH)}}\right] = \frac{1}{|\Lambda_N|}\log \mathbb{E}_N\left[\mathrm{e}^{-|\Lambda_N|H^{(PCH)}}\right] + \frac{1}{|\Lambda_N|}\log \mathbb{E}_N^{(PCH)}\left[\mathrm{e}^{|\Lambda_N|\frac{b}{2}\sum_{k\in A_N}k^2\left(\boldsymbol{\lambda}_k^{(N)}\right)^2}\right] \tag{6.3.56}$$

$$\geq \frac{1}{|\Lambda_N|}\log \mathbb{E}_N\left[\mathrm{e}^{-|\Lambda_N|H^{(PCH)}}\right] + \frac{b}{2}\mathbb{E}_N^{(PCH)}\left[\sum_{k\in\mathbb{N}\setminus A_N}k^2\left(\boldsymbol{\lambda}_k^{(N)}\right)^2\right]. \tag{6.3.57}$$

Now note that that $\mathbb{N}\setminus A_N$ states in the PCH model only interact with each other via the total particle number. Therefore we can repeat arguments like those in Chapter 5 to show that the occupation density of each cycle type remains order 1 (specifically $q_k\mathrm{e}^{\eta k}$ for some $\eta \leq 0$). Now taking the limit proves our inequality.

□

6.4 Variational and Condensate Analysis

In this section we study the variational expressions arising from our large deviation analysis, in particular we study the condensate behaviour. First we consider the cycle-space model, before we repeat this for the momentum space model. This later analysis is largely similar, with simple adjustments made for the appearance of λ_0 and λ_+.

6.4.1 Cycle-Space Analysis

In this subsection, we only consider the $\min A_N \ll |\Lambda_N|$ case. If we were to be considering $\min A_N \gg |\Lambda_N|$, then we would essentially be forbidding our b counter-terms from having any influence. It would essentially be the PM model.

First we consider non-positive values of b. In the momentum picture, the FMH model with a positive counter-term has been investigated by [San04], [MV99], and [Sch90]. In particular, they find that the FMH thermodynamic pressure is equal to the mean field thermodynamic pressure, and here we show that this also holds our partial cycle-space version.

Theorem 6.4.1. *Suppose $b \leq 0$. Then for all $\mu \in \mathbb{R}$,*

$$p^{(PCH)}(\beta,\mu) = p^{(PM)}(\beta,\mu) = \sup_{x\geq 0}\left\{\mu x - \frac{a}{2}x^2 - f(\beta,x)\right\}. \tag{6.4.1}$$

Proof. Let us denote $g : [0,+\infty) \to \mathbb{R}$,

$$g(x) = \mu x - \frac{a}{2}x^2 + \frac{(\mu - ax)_+^2}{2(a-b)} \tag{6.4.2}$$

$$= \begin{cases} \frac{b}{a-b}\left(\frac{\mu^2}{2b} - \mu x + \frac{a}{2}x^2\right) & : 0 \leq x \leq \frac{\mu}{a} \\ \mu x - \frac{a}{2}x^2 & : x \geq \frac{\mu}{a}. \end{cases} \tag{6.4.3}$$

Then $p^{(PCH)}(\mu) = \sup_{x\geq 0}\{g(x) - f(x)\}$ from Theorem 6.2.2.

First suppose $\mu \geq 0$. Note that for $b \leq 0$ the function g is concave and has a maximum at $x = {}^{\mu}/_{a}$. Also, the convex free energy f is strictly convex on $x \in [0, \varrho_c]$ and constant for $x \geq \varrho_c$. Therefore if $\mu \geq a\varrho_c$ then $g - f$ is maximised at $x = {}^{\mu}/_{a}$, and if $\mu < a\varrho_c$ then $g - f$ is maximised by some $x \in ({}^{\mu}/_{a}, \varrho_c)$. In both cases, we have the maximiser satisfies $x \geq {}^{\mu}/_{a}$, and therefore evaluating $g - f$ at this x gives the required form.

For $\mu < 0$, g is strictly concave and has a maximum at $x = 0$. Then the previous argument follows similarly and $g - f$ is maximised by some $x \in (0, \varrho_c)$.

□

To get a physical heuristic on why the PM and the PCH models (for $b \leq 0$) give the same thermodynamic pressure, consider qualitatively what behaviour the PM model and the new counter-term encourage. For a given overall density, the PM model likes to have the cycles spread out over the different types according to the weights q (with some adjusted chemical potential), and the counter-term also likes to have the particles spread out over the cycle types (albeit more uniformly). Therefore they are both working towards similar aims, and as $\min A_N \to \infty$ any effect of the counter-term vanishes. In contrast, if we have $b > 0$ then the counter-term encourages the particles in the affected cycles to congregate in a single cycle type,

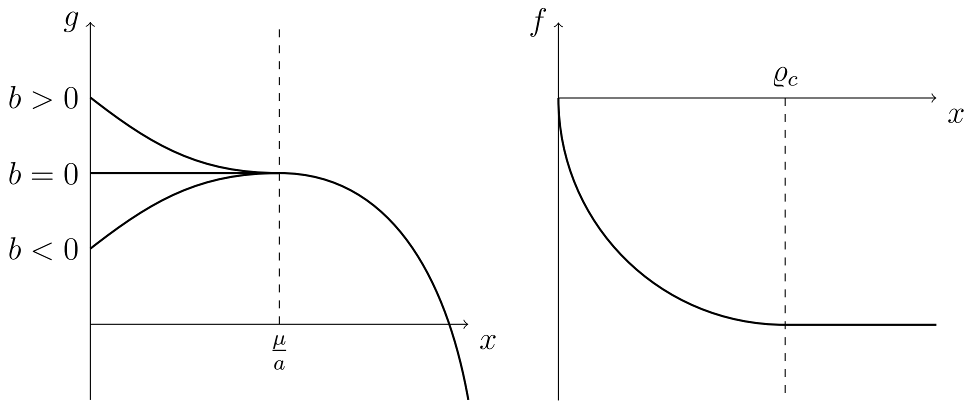


Figure 6.2: The function g (for $\mu > 0$) and the free energy f (for $d \geq 3$) as functions of x.

and is working contrary to the efforts of the PM model. It is therefore conceivable that qualitative overall behaviour persists in the thermodynamic limit, and we show here that this is in fact the case for some region of the parameter-space.

For the following results, we define the functions $s_\beta : (0, \infty) \to \mathbb{R}$ by

$$s_\beta : x \mapsto \begin{cases} 0 & : x \geq \varrho_c(\beta) \\ \text{unique solution to } p'(\beta, s) = x & : 0 < x \leq \varrho_c(\beta) . \end{cases} \tag{6.4.4}$$

Here and hereafter we let $p'(\beta, s)$ and $f'(\beta, x)$ denote the s and x partial derivatives respectively. Note that $f(\beta, x) = s_\beta(x)\, x - p(\beta, s_\beta(x))$, that $f'(\beta, x) = s_\beta(x)$, and that s_β is concave everywhere and strictly concave for $x \leq \varrho_c$. We plot examples of s_β for $\varrho_c < \infty$ in Figure 6.3. In what follows we shall also encounter qualitatively different behaviour in our model, depending upon the value of $\lim_{\epsilon\downarrow} p''(\beta, -\epsilon)$ (allowing $= +\infty$, as it diverges for $d \leq 4$). For clarity we will abuse notation and call this limit $p''(\beta, 0)$, even though p is not right differentiable in s at $s = 0$.

Lemma 6.4.2. *Let $b > 0$. Let $\mu_t(\beta) = \inf_{u<0} \left\{ a p'(\beta, u) - \frac{a-b}{b} u \right\} > 0$ and $\mu_c = a\varrho_c$. Then there exists a $\mu^* \in [\mu_t, \mu_c]$ such that $I_\alpha + \beta \underline{H}$ is minimised by $(\tilde{x}, \tilde{y})$.*

$$\tilde{x} = \begin{cases} x_1 & : \mu \leq \mu^* \\ x_2 & : \mu \geq \mu^* \end{cases}, \qquad \tilde{y} = \frac{(\mu - a\tilde{x})_+}{a - b}, \tag{6.4.5}$$

where

$$x_1 \text{ is the unique solution to } s_\beta(x) = \beta(\mu - ax) \tag{6.4.6}$$

$$x_2 \text{ is the minimal solution to } s_\beta(x) = -\frac{b}{a-b}\beta(\mu - ax). \tag{6.4.7}$$

If $\mu^* = \mu_c$, we always have a unique minimiser. Otherwise we have a unique minimiser if $\mu \neq \mu^*$ and precisely two global minimisers if $\mu = \mu^*$.

Furthermore,

$$p''(\beta, 0) \leq \frac{a-b}{ab} \iff \mu^* = \mu_c, \tag{6.4.8}$$

and

$$p^{(PCH)}(\beta, \mu) = \begin{cases} \inf_{s<0}\left\{\frac{(\mu-s)^2}{2a} + p(\beta, s)\right\} & : \mu \leq \mu^* \\ \sup_{s<0}\left\{\frac{(\mu-s)^2}{2a} - \frac{s^2}{2b} + p(\beta, s)\right\} & : \mu \geq \mu^*. \end{cases} \tag{6.4.9}$$

PROOF. This proof is closely related to the proof of [BLP88, Theorem 2]. We formulate it in terms of the density rather than chemical potentials so that we more directly see the global minimiser. First we fix x, and find the optimal choice of y. Since f is y-independent, we easily find that $\tilde{y} = \frac{(\mu-ax)_+}{a-b}$ by considering the y-partial derivative.

For notational ease, we now denote

$$F(x) = (I_\alpha + \underline{H})(x, \tilde{y}(x)) = f(x) - g(x), \tag{6.4.10}$$

$$t(x) = g'(x) = \begin{cases} \mu - ax & : x \geq \frac{\mu}{a} \\ -\frac{b}{a-b}(\mu - ax) & : x \leq \frac{\mu}{a}. \end{cases} \tag{6.4.11}$$

Now $\frac{\mathrm{d}F}{\mathrm{d}x} = s(x) - t(x)$ is defined for all $x > 0$. Furthermore, $s, t \leq 0$, $\lim_{x\downarrow 0} s(x) = -\infty$, and $\lim_{x\to\infty} t(x) = -\infty$, and therefore the global minimum is achieved by a solution to

$$s(x) = t(x). \tag{6.4.12}$$

For $\mu > \mu_c = a\varrho_c$, we have two stationary points - $x_1 = {}^{\mu}/_{a} > \varrho_c$ and $x_2 < \varrho_c$. However, at x_1 the function f is a constant and g is at an inflection. Therefore x_1 is an inflection and x_2 is the global minimum.

Now we consider $\mu \leq \mu_c$. Since s is increasing, and $x \mapsto \mu - ax$ is strictly decreasing, x_1 always exists and exists uniquely. However, this is not the case for x_2. $x \mapsto -\frac{b}{a-b}\left(\mu - ax\right)$ is an increasing affine function, and s is concave, so for $\mu > \mu_t$ there are two solutions to $s\left(x\right) = -\frac{b}{a-b}\left(\mu - ax\right)$, these coincide at $\mu = \mu_t$, and no solution exists for $\mu < \mu_t$. This is 'proven by picture' in Figure 6.3. We denote the larger of these solutions x_3 and the smaller x_2.

For $\mu < \mu_t$ we have a unique stationary point, x_1, and this is therefore the global minimum.

For $\mu \in [\mu_t, \mu_c]$, we first eliminate x_3 from our considerations. At $\mu = \mu_t$, $x_2 = x_3$, and at $\mu = \mu_c$, $x_3 = x_1$. For $\mu \in (\mu_t, \mu_c)$, we see that $\frac{\mathrm{d}F}{\mathrm{d}x}\left(x_3 - \delta\right) > 0$ and $\frac{\mathrm{d}F}{\mathrm{d}x}\left(x_3 + \delta\right) < 0$ for sufficiently small $\delta > 0$, and therefore x_3 is a local maximum. We now need to compare x_1 and x_2 for $\mu \in [\mu_t, \mu_c]$. For $i = 1, 2$, denote

$$F_i\left(\mu\right) = F\left(x_i\left(\mu\right)\right). \tag{6.4.13}$$

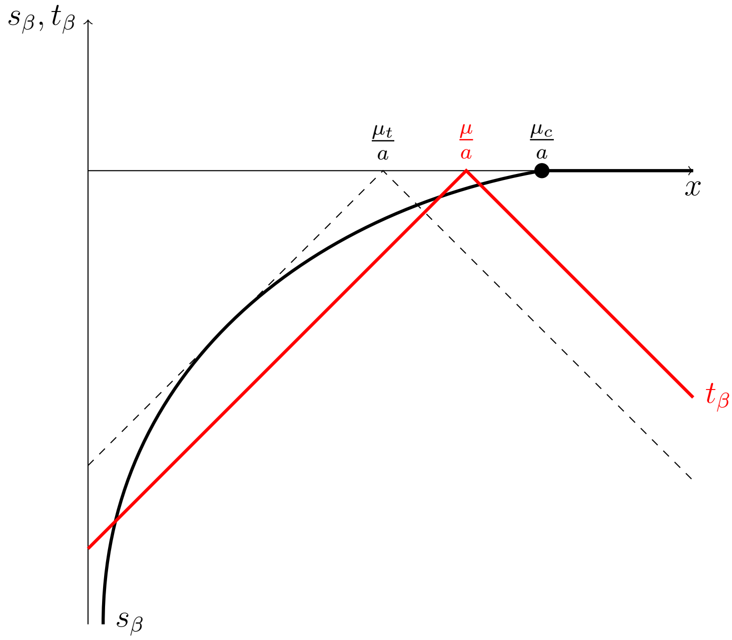
Now since x_i are stationary points, we easily get

$$\frac{\mathrm{d}F_i}{\mathrm{d}\mu} = -x_i - \frac{\left(\mu - ax_i\right)_+}{a-b} = \begin{cases} -x_1 & : i = 1 \\ \frac{bx_2 - \mu}{a-b} & : i = 2 \end{cases} = \begin{cases} \frac{s(x_1)-\mu}{a} & : i = 1 \\ \frac{s(x_2)-\mu}{a} & : i = 2. \end{cases} \tag{6.4.14}$$

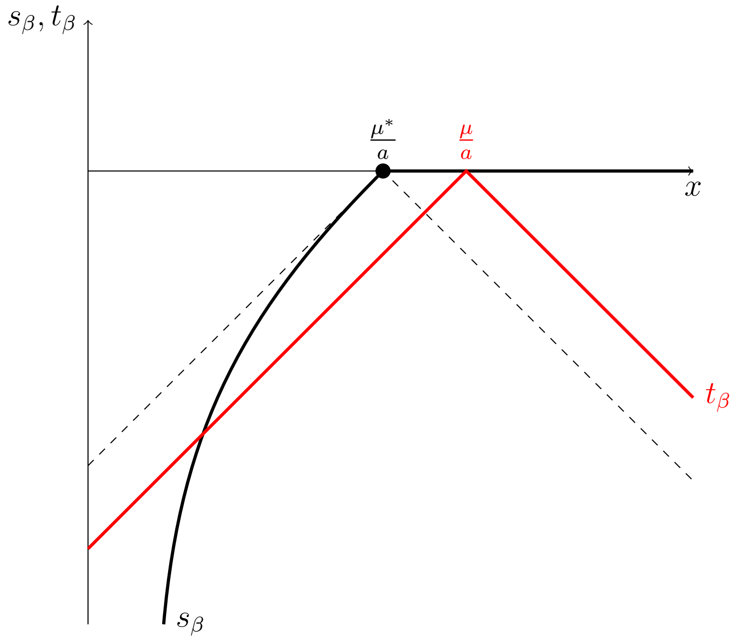
Since s is an increasing function (strictly increasing on the relevant region), we have $\frac{\mathrm{d}}{\mathrm{d}\mu}\left(F_1 - F_2\right) = \frac{1}{a}\left(s\left(x_1\right) - s\left(x_2\right)\right) > 0$ for $\mu \in (\mu_t, \mu_c)$. Furthermore, since x_1 merges with x_3 at $\mu = \mu_c$ and x_2 merges with x_3 at $\mu = \mu_t$, these are inflection points at these respective points. Therefore we have $F_1\left(\mu_t\right) < F_2\left(\mu_t\right)$ and $F_1\left(\mu_c\right) > F_2\left(\mu_c\right)$, and there exists a unique $\mu^* \in (\mu_t, \mu_c)$ at which the global minimum switches from x_1 to x_2. Even when $\mu_c = +\infty$ (i.e. $d = 1, 2$), note that $\frac{\mathrm{d}}{\mathrm{d}\mu}\left(F_1 - F_2\right)$ is increasing in μ (since x_1 is increasing and x_2 is decreasing), and therefore the switch does eventually happen.

In particular, this argument implies that $\mu^* < \mu_c$ if and only if $\mu_t < \mu_c$. By considering the derivative of $u \mapsto ap'\left(u\right) - \frac{a-b}{b}u$ for $u < 0$, we find that $\mu_t = \mu_c$ (and therefore $= \mu^*$) if and only if $p''\left(0\right) \leq \frac{a-b}{ab}$. Note that this is only possible if $d \geq 5$.

To prove (6.4.9), we work backwards from the results, and find that the



(a) $\mu_t < \mu_c$



(b) $\mu_t = \mu_c$

Figure 6.3: Figures demonstrating how μ_t and μ_c can coincide and separate. The value t for $\mu = \mu_t$ is plotted in dashes.

optimal s satisfies the conditions we require of $s(x_i)$ for the relevant i.

$\square$

Corollary 6.4.3. *Fix $\beta > 0$. For $\mu \le \mu^*(\beta)$,*

$$p^{(PCH)}(\beta,\mu) = p^{(PM)}(\beta,\mu). \tag{6.4.15}$$

For $\mu > \mu^*(\beta)$,

$$p^{(PCH)}(\beta,\mu) > p^{(PM)}(\beta,\mu). \tag{6.4.16}$$

PROOF. We prove this be looking at the minimisers from Lemma 6.4.2 and the form of the pressure in Theorem 6.2.2. For $\mu \le \mu^*$, $\tilde{y} = 0$ and so the expression is clearly equal to the PM model (set $b = 0$). For $\mu > \mu^*$, we have found that we can optimise by moving off the line $y = 0$, and so the PCH pressure is greater.

$\square$

Lemma 6.4.4. *Let $b > 0$ and suppose $(\tilde{x},\tilde{y})$ is the unique minimiser of $\mathcal{J}_\alpha + \beta\underline{H}$. Then for any $\delta > 0$,*

$$\lim_{N\to\infty} \mathsf{Q}_N^{(PCH)}\left(\left|M_1^{(A,N)} - \tilde{x}\right| < \delta \text{ and } \left|M_2^{(A,N)} - \tilde{y}\right| < \delta\right) = 1. \tag{6.4.17}$$

PROOF. This follows from the first part of Lemma 6.2.9. Note that the product of open balls $O = B_{\tilde{x}}^\delta \times B_{\tilde{y}}^\delta$ is open. Therefore $O^c \subset [0,+\infty]^2$ is closed and

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \int_{\left(\pi_N^{(A)}\right)^{-1}(O^c)} \mathrm{e}^{-H_N^{(PCH)}} \mathsf{Q}_\alpha(\mathrm{d}\omega) \le -\inf_{O^c}\left\{\mathcal{J}_\alpha + \underline{H}\right\}. \tag{6.4.18}$$

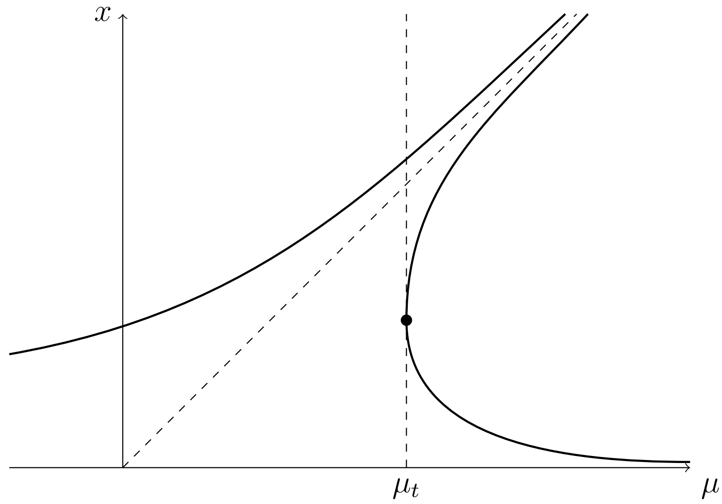
Since $(\tilde{x},\tilde{y}) \in O$ is the unique minimiser,

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathsf{Q}_N^{(PCH)}\left(\left(\pi_N^{(A)}\right)^{-1}(O^c)\right) \le \inf_{[0+\infty]^2}\left\{\mathcal{J}_\alpha + \underline{H}\right\} - \inf_{O^c}\left\{\mathcal{J}_\alpha + \underline{H}\right\} < 0. \tag{6.4.19}$$

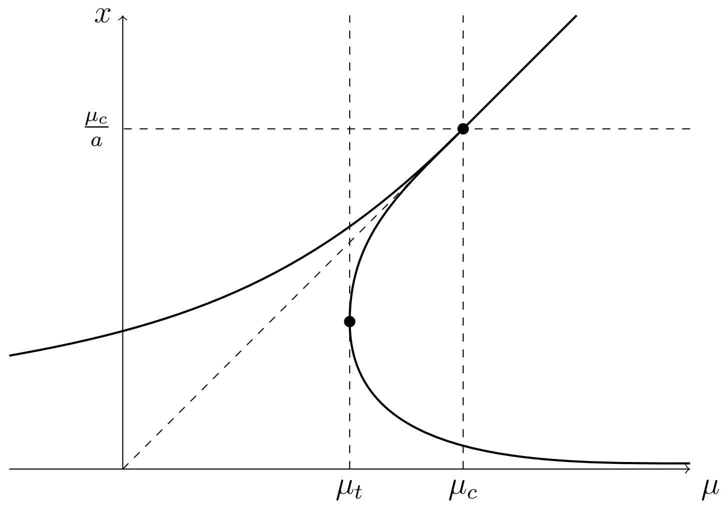
Therefore $\lim_{N\to\infty} \mathsf{Q}_N^{(PCH)}\left(\left(\pi_N^{(A)}\right)^{-1}(O^c)\right) = 0$, and the result follows.

$\square$

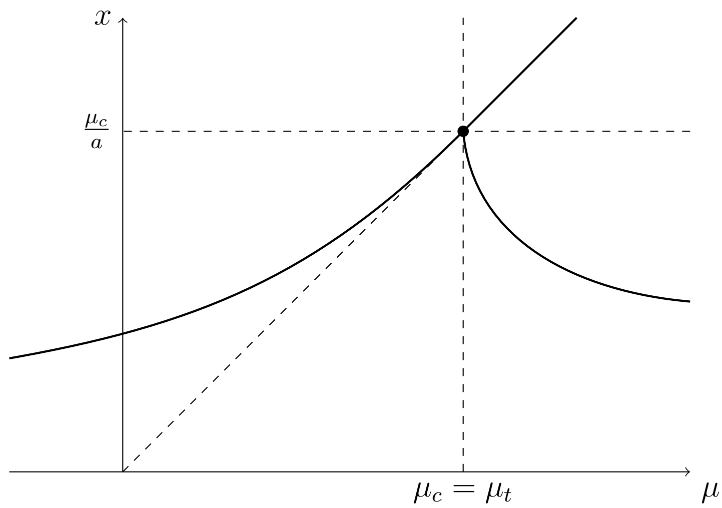
Remark 6.4.5. *We can compare this with the mean-field results of [BCMP05, Theorem 3.2]. They consider sets A_N of the form $A_N = \{J(L_N), J(L_N)+1, \dots\}$ where*



(a) $d = 1, 2$



(b) $d \geq 3$ and $p''(\beta, 0) > \frac{a-b}{ab}$



(c) $d \geq 3$ and $p''(\beta, 0) \leq \frac{a-b}{ab}$ (implies $d \geq 5$)

Figure 6.4: Value of x at stationary points of $I_\alpha + \beta \underline{H}$.

$|\Lambda_N| = L_N^d$ *and* $J(L) \leq L^2$. *In dimensions* $d \geq 3$ *we have a weaker condition on* A_N*, but this is because our interaction energy specifically encourages particles to be in the* A_N*-states. We cannot apply the argument of Lemma 6.4.4 to the condensate regimes of the PM model* ($b = 0$)*, because for* $\mu > a\varrho_c$ *every convex combination of* $\left(\frac{\mu}{a}, 0\right)$ *and* $\left(\varrho_c, \frac{\mu}{a} - \varrho_c\right)$ *is a global minimiser of*

$$f(\beta, x) - \mu(x + y) + \frac{a}{2}(x + y)^2. \tag{6.4.20}$$

$\diamond$

Remark 6.4.6. *The following two results (Corollaries 6.4.7 and 6.4.8) can be proven by arguments like that in [BLP88, Theorem 3] and in our proofs in Section 5.3 . In this type of argument we add an extra bit of chemical potential that only affects the particles we are interested in (the* A_N *states or states above the cut-off* K*) and find the resulting thermodynamic (pseudo-)pressure using the techniques of Section 6.2. We then use Griffith's Lemma to match the derivative of this limit (with respect to this extra chemical potential) with the limit of the derivatives of the finite systems. In fact, these derivatives of the finite systems give the expected particle number contained in the relevant states, and therefore we have our result. Here we instead present the results as a consequence of our wider convergence-in-probability result, Lemma 6.4.4.* $\diamond$

Corollary 6.4.7. *If* $\mu \neq \mu^*(\beta)$ *or* $p''(\beta, 0) \leq \frac{a-b}{ab}$*, the* A_N*-condensate density is given by*

$$\Delta_A^{(\text{PCH})}(\beta, \mu) = \tilde{y}. \tag{6.4.21}$$

Furthermore,

$$p''(\beta, 0) \leq \frac{a-b}{ab} \iff \Delta_A^{(\text{PCH})}(\beta, \mu) \text{ is continuous.} \tag{6.4.22}$$

Proof. This is a direct corollary of Lemma 6.4.4, with the unique minimisers found by Lemma 6.4.2. For the continuity of the condensate, note that $x_1(\mu^*) > x_2(\mu^*)$ if $\mu^* < \mu_c$, and $x_1(\mu^*) = x_2(\mu^*)$ if $\mu^* = \mu_c$.

$\square$

Corollary 6.4.8. *If* $\mu \neq \mu^*(\beta)$ *or* $p''(\beta, 0) \leq \frac{a-b}{ab}$,

$$\Delta^{(\text{PCH})}(\beta, \mu) = \Delta_A^{(\text{PCH})}(\beta, \mu). \tag{6.4.23}$$

PROOF. This is a less natural corollary, and we have to work a little harder. We begin by introducing the random variables

$$M_1^{(N,K)} = \sum_{j \leq K} j \boldsymbol{\lambda}_j^{(N)}, \qquad M_2^{(N,K)} = \sum_{j > K} j \boldsymbol{\lambda}_j^{(N)}, \tag{6.4.24}$$

and $\pi_N^{(K)}, Q_{N,\alpha}^{(K)}$ analogous to the $\pi_N^{(A)}, Q_{N,\alpha}^{(A)}$ defined before. By repeating the arguments of Lemmas 6.2.6 and 6.2.7, we can show that for $\alpha < 0$ the distributions $Q_{N,\alpha}^{(K)}$ obey a LDP on $\mathbb{R}_+^2$ with rate $|\Lambda_N|$ and rate function

$$\mathcal{J}_\alpha^{(K)}(x, y) = f_K(x, y) - \alpha(x + y) + p(\alpha), \tag{6.4.25}$$

where

$$f_K(x, y) = \sup_{s \in \mathbb{R}} \left\{ sx - \sum_{j \leq K} q_j e^{sj} \right\} + \sup_{t \leq 0} \left\{ ty - \sum_{j > K} q_j e^{tj} \right\}. \tag{6.4.26}$$

Now, since $A_N \subset \{K+1, K+2, \ldots\}$ eventually, we have $H_N^A(\omega) \geq \underline{H} \circ \pi_N^K(\omega)$ and we get the result analogous to Lemma 6.2.9. Since we already have the thermodynamic pressure, we also get the result analogous to Lemma 6.4.4. Since $\underline{H}$ is a constant plus a positive definite quadratic form, and $f_K((x, y)$ is convex and bounded below by $f(x+y)$, we know that there are only finitely many global minimisers, say $\left\{\left(x_i^K, y_i^K\right)\right\}_{i \in I_K}$, and that these are contained in some K-independent compact set. This bound on f_K holds because $f(x+y)$ is defined as the same supremum as that in $f_K(x, y)$ but with the constraint that $s = t$. Now let $\hat{B}_K$ be the set of convex combinations of these minimisers, and $\hat{B}_K^\delta$ be the δ-'ball' of this set:

$$\hat{B}_K^\delta = \left\{ z \in \mathbb{R}^2 : \exists z' \in \hat{B}_K \text{ s.t. } |z - z'| < \delta \right\}. \tag{6.4.27}$$

We therefore have

$$\lim_{N\to\infty} \mathtt{Q}_N^{(\text{PCH})}\left(\hat{B}_K^{\delta}\right) \to 1, \qquad \forall\delta > 0. \tag{6.4.28}$$

To prove our result we now only need to show that these finite K minimisers converge to $(\tilde{x}, \tilde{y})$. Precisely, given $\delta > 0$, there exists a K' such that $K > K'$ implies that $\bigcup_{i\in I_K}\left\{\left(x_i^K, y_i^K\right)\right\} \subset B^{\delta}_{(\tilde{x},\tilde{y})}$. We do this by proving uniform convergence on our K-independent compact set.

Note that $\sum_{j>K} q_j \mathrm{e}^{tj} \le \sum_{j>K} q_j$ on $t \le 0$, and hence

$$\left|\sup_{t\le 0}\left\{ty - \sum_{j>K} q_j \mathrm{e}^{tj}\right\}\right| \le \sum_{j<K} q_j \to 0. \tag{6.4.29}$$

For ease of notation, let

$$p_K(s) = \sum_{j\le K} q_j \mathrm{e}^{sj}. \tag{6.4.30}$$

For $x < \varrho_c$, the relevant supremum is eventually achieved on $s < 0$. Like for the t-supremum, we have uniform convergence $p_K(s) \to p(s)$ on $s \le 0$ and $f_K(x, y) \to f(x)$ uniformly on $x < \varrho_c$. For $x \ge \varrho_c$ we note that p_K is analytic, with all derivatives being positive. In particular, for $n \ge 2$ we have

$$sx - p_K(s) \le sx - p_K(0) - sp_K'(0) - \frac{1}{n!}s^n p_K^{(n)}(0). \tag{6.4.31}$$

The right hand side is a polynomial in s, and is easily maximised. In particular,

$$sx - p_K(s) \le -p_K(0) + C_{n,d}\frac{1}{\left(p_K^{(n)}(0)\right)^{\frac{1}{n-1}}}\left(x - p_K'(0)\right)^{1+\frac{1}{n-1}}. \tag{6.4.32}$$

For $n \ge d/2$, we have $p_K^{(n)}(0) \sim c_{n,d}\log K$ for some $c_{n,d} > 0$. In particular, $p_K^{(n)}(0) \to \infty$ as $K \to \infty$. Since $p_K'(0) \to \varrho_c$, we have

$$f_K(x, y) \le f(x) + \varepsilon_K^{(1,\delta)} + \varepsilon_K^{(2,\delta)}\left(x - \varrho_c\right)_+^{1+\delta}, \tag{6.4.33}$$

for any $\delta > 0$, where $\epsilon_K^{(i,\delta)} \to 0$. Whilst we don't have uniform convergence on $\mathbb{R}_+^2$, we have uniform convergence on any compact set as required.

$\square$

Theorem 6.4.9. *Fix $\beta > 0$ and $d \geq 1$.*

- *Let $a \to \infty$ with fixed $\mu \in \mathbb{R}$ and ratio $a/b \in (1, +\infty)$. Then*

$$\Delta(\beta, \mu) = 0 \text{ eventually.} \tag{6.4.34}$$

- *Let $b \downarrow 0$ with fixed $\mu \in \mathbb{R}$ and $a > 0$. Then*

$$\Delta(\beta, \mu) \to \left(\frac{\mu}{a} - \varrho_c(\beta)\right)_+. \tag{6.4.35}$$

- *Let $b \uparrow a$ with fixed $\mu \in \mathbb{R}$ and $a > 0$. Then*

$$\Delta(\beta, \mu) \sim \left(\frac{\mu}{a-b}\right)_+. \tag{6.4.36}$$

- *Let $\mu \to \infty$ with fixed $a > b > 0$. Then*

$$\Delta(\beta, \mu) \sim \frac{\mu}{a-b}. \tag{6.4.37}$$

- *Let $\mu \to \infty$ with fixed ratios ${}^{\mu}\!/_{a}$ and ${}^{\mu}\!/_{b}$. Then*

$$\Delta(\beta, \mu) \to \frac{\mu}{a-b}. \tag{6.4.38}$$

Proof. For case (6.4.34), we have $\mu_t \to \infty$, and so eventually $\mu < \mu_t \leq \mu^*$.

For case (6.4.35), note that $x_2 \to \varrho_c$ if $\mu > \mu_c$, and thus $\Delta \to \frac{\mu}{a} - \varrho_c$. Since Δ is necessarily non-increasing in μ, this gives the asymptotics for all μ.

For case (6.4.36), note first that $\mu_t \to 0$ and $x_2 \to 0$, whilst x_1 is b-independent and greater than ${}^{\mu}\!/_{a}$. Therefore $F(x_2) \sim \frac{-\mu^2}{2(a-b)}$, whilst $F(x_1)$ is constant. Hence x_2 is eventually preferred for any $\mu > 0$, and $\Delta \sim \frac{\mu}{a-b}$.

For case (6.4.37), we eventually only have the one local extremum, that corresponding to x_2. Since $x_2 \to 0$, we have the result.

For case (6.4.38), of the terms in F, only f remains fixed whilst all the others grow at the same rate. Hence as long as the extrema remain in a bounded region, we only need to look at these other terms. Furthermore, f decays slower than linearly, so this is indeed the case. The result follows from considering these other terms.

□

6.4.2 Momentum-Space Analysis

Whilst our pressure expression for the momentum space model is essentially the same as for our cycle-space model, we present the theorems here in a similar style to [BLP88] to emphasise the roles played by a, b, λ_0, and λ_+. Note that the derivative $p'(\beta,\alpha)$ acts on the the chemical potential:

$$p'(\beta,\alpha) := \frac{\partial}{\partial s} p(\beta,s)\bigg|_{s=\alpha}. \tag{6.4.39}$$

Theorem 6.4.10. *Let* $\mu_t = \inf_{\alpha<0}\left\{ap'(\beta,\alpha) - \frac{a-b}{b}\alpha\right\}$ *and* $\mu_c = a\varrho_c$. *Then there exists a unique* $\mu^* \in [\mu_t, \mu_c]$ *such that*

$$p^{(PMH)}(\mu) = \begin{cases} \inf_{\alpha<0}\left\{\frac{(\mu+\lambda_+-\alpha)^2}{2a} + p(\beta,\alpha)\right\} & : \mu+\lambda_+ \le \mu^* \\ \sup_{\alpha<0}\left\{\frac{(\mu+\lambda_+-\alpha)^2}{2a} - \frac{\alpha^2}{2b} + p(\beta,\alpha)\right\} & : \mu+\lambda_+ \ge \mu^*. \end{cases} \tag{6.4.40}$$

Equivalently,

$$p^{(PMH)}(\beta,\mu) = \max\left\{\frac{(\mu+\lambda_+-\alpha_1)^2}{2a} + p(\beta,\alpha_1), \frac{(\mu+\lambda_+-\alpha_2)^2}{2a} - \frac{\alpha_2^2}{2b} + p(\beta,\alpha_2)\right\}, \tag{6.4.41}$$

where α_2 *is the the most negative solution to* $ap'(\beta,\alpha) = \frac{a-b}{b}\alpha + \mu + \lambda_+$ *(exists for* $\mu+\lambda_+ \ge \mu_t$*), and* α_1 *is the unique solution to* $ap'(\beta,\alpha) = -\alpha + \mu + \lambda_+$ *(exists for* $\mu + \lambda_+ \ge \mu_c$*).*

Furthermore,

$$p''(\beta,0) \le \frac{a-b}{ab} \iff \mu^* = a\varrho_c. \tag{6.4.42}$$

Proof. Is the proof of [BLP88, Th. 2], keeping track of the parameters a, b, λ_0, and λ_+.

$\square$

For the momentum space model, the generalised condensate of [Gir60] can be written as

$$\Delta^{(PMH)}(\beta,\mu) = \lim_{\lambda \downarrow 0} \lim_{N\to\infty} \mathbb{E}^{(PMH)}_{\mu,N}\left[\frac{1}{|\Lambda_N|}\sum_{j:\lambda_j^{(N)}\leq\lambda}\sigma_j\right]. \tag{6.4.43}$$

This served as the inspiration for our condensate densities in the cycle space models.

Theorem 6.4.11. *If F is continuous the total amount of condensate $\Delta^{(PMH)}(\beta,\mu)$ is given by*

$$\Delta^{(PMH)}(\beta,\mu) = \begin{cases} -\frac{\alpha(\beta,\mu)}{b} & : \mu+\lambda_+ > \mu^* \\ 0 & : \mu+\lambda_+ < \mu^*, \end{cases} \tag{6.4.44}$$

where $\alpha(\beta,\mu)$ is the the most negative solution to $ap'(\beta,\alpha) = \frac{a-b}{b}\alpha + \mu + \lambda_+$.

Furthermore,

$$\lim_{\varepsilon\downarrow 0} p''(\beta,-\varepsilon) \leq \frac{a-b}{ab} \iff \Delta^{(PMH)}(\beta,\mu) \text{ is continuous.} \tag{6.4.45}$$

Proof. This is the proof of [BLP88, Th. 3], keeping track of the parameters a, b, λ_0, and λ_+.

$\square$

Chapter 7

Conclusion and Topics of Future Study

In this thesis we have studied three different resolutions of the Bose gas random loop soup. First we studied the full empirical stationary measure, proving a LDP for positive interactions and for the full parameter space with stabilisation. However the rate function of the non-interacting model is not as explicit as we have for our cycle models. This means that, for example, we don't have as good an understanding of the minima of the interacting rate functions - in particular their uniqueness. This means that the techniques used later in this thesis cannot yet be applied to this high-resolution model.

We also discussed the progress made in describing models beyond the condition that the pair-potential is non-negative. Having negative energy interactions is often a difficulty in the study of Bose gases. The canonical ensemble dilute Bose gas in 3-dimensions is discussed by [LSSY05]. They prove that the thermodynamic limit of the energy density of the ground state, $e_0(\varrho)$, has the following asymptotics:

$$\lim_{\varrho a^3 \to 0} \frac{e_0(\varrho)}{4\pi\varrho a} = 1, \tag{7.0.1}$$

where a is the scattering length of the interaction potential v. Their proof assumes that v is non-negative, but they conjecture that (7.0.1) also holds as long as $a > 0$ and v has no N-body bound states for any N. This condition is a very complex one: [Bau97] proves that there are potentials with positive scattering length and no 2-body bound states, that nevertheless do have 3-body bound states. If we are hoping to be able to describe a Lennard-Jones type interaction, we expect that we will encounter some similarly complex conditions on the interaction.

One way that a negative interaction could cause a break-down on the Bose gas would be that the particle density diverges. If the interaction has a negative component, then there could exist configurations in which we can keep on adding particles and arriving at lower and lower interaction energy levels. For fermions we have the Pauli exclusion principle that prevents the particle density from exploding in this manner but no such inherent property exists for bosons. One way to avoid this break-down is to incorporate a hard-core into the interaction to ensure the particle density cannot keep increasing, and our analysis allows us to consider such a scenario either by directly applying a hard-core via the interaction potential, or by restricting the configuration space and using Lemma 3.1.7.

In Chapter 4 we took a step back and derived a LDP for the empirical cycle measure under a number of interaction energies. Whilst this had the cost of losing the details of the spatial behaviour of the loop marks and the pair-potential energy, the ideal gas rate function is given explicitly and the topological space ℓ_1 is well-understood. The CM model is a new model that behaves relatively simply, whilst the PM model has been extensively studied and is well-understood. The major novelty here is the introduction of - and study of - the full cycle HYL (FCH) model. This FCH model cannot be described in terms of the empirical particle total density, like the ideal and PM models can, because the interaction energy depends upon on how the particle mass is distributed amongst the cycle types. Because we are working in our cycle space, and have the corresponding ℓ_1-topology, the PM and FCH interaction energy is non-continuous and proving the LDP is non-trivial. Furthermore, they are neither purely lower nor upper lower semicontinuous, and so we do not even get one of the large deviation bounds for free. One other standard approach to studying discontinuous interaction energies is to apply a cut-off, say in the maximum cycle length. However, neither the PM nor the FCH interactions would be monotone as we would remove this cut-off, so this technique also has complications. We are able to overcome these difficulties by paying particular attention to the lower ℓ_1-semicontinuous regularisations of our interaction energies. Because we have our explicit expression for the non-interacting model rate function, we are able to more easily study the variational expressions we get and show that it is in fact lower semicontinuous in the same way that our regularised interactions energies are.

In contrast to the rate functions we get for the stationary empirical measures, the rate functions for the empirical cycle measures are very explicit. This makes it much more feasible to study their minimisers. This allows us, in Chapter 5, not only to find convergence results to these minima, but to prove thermodynamic pressure limits and evaluate condensate densities. These condensate densities can then act as order parameters for BEC.

The most obvious avenue for progress here is to continue to explore the minima behaviour for the FCH model. We have shown that there are arbitrarily many stationary points (found by increasing the chemical potential to arbitrarily high values), and these are not necessarily easily compared. There are also qualitatively different behaviours depending upon the dimension d, the inverse temperature β, and the coupling parameter b.

One issue with our minima of the ℓ_1 rate functions is that we only have convergence in the cycle count. Since the particle density is not continuous in the cycle count sequences, these don't immediately tell us how the particle numbers behave. In Chapter 6, we tried to overcome this by deriving LDP on $\mathbb{R}^2_+$ directly for the particle numbers associated with two classes of cycle types. This has the additional reward that we are able to elegantly prove the pressure for the PCH model by using these techniques. This PCH model also has a much simpler minimiser behaviour than the FCH model, but nevertheless shows similar patterns in some regimes. Further exploring this relation may allow us to demonstrate that the FCH model exhibits more than one condensation phenomena - BEC may be characterised by jumps at long cycles, whilst jumps at short cycles may be evidence of a different transition.

In Chapter 6 we also restricted the PCH models we considered to those with

$\min A_N \ll |\Lambda_N|$ and those with $\min A_N \gg |\Lambda_N|$. This of course leaves a gap: what happens if there exist $0 < c_1 < c_2 < +\infty$ such that ${}^{\min A_N}/_{|\Lambda_N|} \in (c_1, c_2)$ eventually? It is clear that some A_N-particle-densities are impossible for sufficiently large N - the only permitted density below c_1 would eventually be 0. The proof of a large deviation principle or thermodynamic pressure is therefore going to rely heavily upon the permitted values of the density - a combinatorial question. One case in this regime that may be more manageable is $A_N = \{n \in \mathbb{N} : n \geq m_N\}$, where ${}^{m_N}/_{|\Lambda_N|} \to C \in (0, +\infty)$. In this case it is clear that any density below C cannot be attained eventually, and there are many ways to get any density above C.

One natural continuation of Section 6.3 would be to not just derive the thermodynamic pressure of the FMH model, but to derive a LDP for it as well. This would be a significant improvement on the work of [BLP88]. In [BDLP90a], a further term is added to the HYL energy density. They perturb the interaction energy by the quadratic form

$$\frac{1}{2|\Lambda_N|} \sum_{j,k\in\mathbb{N}} v\left(\lambda_j^{(N)}, \lambda_k^{(N)}\right) \sigma_j(\omega)\, \sigma_k(\omega)\,, \tag{7.0.2}$$

where $v(\cdot,\cdot)$ is a positive-definite continuous function. Deriving a LDP for such a perturbed model would also be an improvement, and it may also be possible to derive a LDP for an analogous model in the cycle-space framework.

Throughout this thesis we have talked about this idea of "long cycles." This is a somewhat messy idea in that we are talking about a limiting object. It would be much nicer if we could expand our sample space to include these as a more explicit object. To do this without it looking shoe-horned in is an ongoing task. In [AFY19], they do this for a slightly different model to the one we consider. Their configuration space consists of a set of points in $\mathbb{R}^d$ (corresponding to the physical particles) and a permutation of these points. BEC is proven for the non-interacting version of this model by [BU11], and they also derive the distribution of "long cycles" in this model. [AFY19] enhance this model by adding infinitely long Brownian interlacements to this model, and showing that a non-trivial distribution for these interlacements corresponds to BEC. It would be very interesting to see if this description fits well into our formulation, and how natural an extension it is. Early questions include asking what the natural topology would be, does the interlacement arise naturally out of finite volume models, and whether our focus on the importance of an anchor point causes complications.

Appendix A

Dirichlet and periodic boundary conditions

A.1 Reference Processes

The following descriptions of the Dirichlet and periodic reference processes are adapted from [ACK11].

Definition A.1.1 (**Dirichlet**). For Dirichlet boundary condition, one restricts the Brownian bridges to not leaving the set $\Lambda \subset \mathbb{R}^d$. Recall that $\mathcal{C}_{1,\Lambda_N}^{(\mathrm{Dir})} \subset \mathcal{C}_1$ denotes the subset of continuous functions $f : [0, \beta] \to \Lambda_N$ satisfying $f(0) = f(\beta)$, equipped with the topology of uniform convergence. On this space we consider the measure

$$\boldsymbol{\mu}_{x,y}^{(\mathrm{Dir},N,\beta)}(A) = \mathbb{P}_x(\mathbb{1}\{B \in A\}\, \delta_y(B_\beta)), \qquad A \subset \mathcal{C}_{1,\Lambda_N}^{(\mathrm{Dir})} \text{ measurable}, \tag{A.1.1}$$

which has total mass

$$g_\beta^{(\mathrm{Dir},N)}(x, y) = \boldsymbol{\mu}_{x,y}^{(\mathrm{Dir},N,\beta)}(\mathcal{C}_{1,\Lambda_N}^{(\mathrm{Dir})}) = \mathbb{P}_x(\mathbb{1}\left\{B_{[0,\beta]} \subset \Lambda_N\right\} \delta_y(B_\beta)). \tag{A.1.2}$$

◇

Definition A.1.2 (**Periodic**). For periodic boundary condition, the marks are Brownian bridges on the torus $\Lambda_N = \left(\mathbb{R}/L_N\mathbb{Z}\right)^d$. Recall that $\mathcal{C}_{1,\Lambda_N}^{(\mathrm{per})}$ denotes the subset of continuous functions $f : [0, \beta] \to \Lambda_N$ satisfying $f(0) = f(\beta)$, equipped with the topology of uniform convergence. The periodic path measure is denoted by $\boldsymbol{\mu}_{x,y}^{(\mathrm{per},N,\beta)}$, and its total mass is equal to

$$g_\beta^{(\mathrm{per},N)}(x,y) = \boldsymbol{\mu}_{x,y}^{(\mathrm{per},N,\beta)}(\mathcal{C}_{1,\Lambda_N}^{(\mathrm{per})}) = \sum_{z\in\mathbb{Z}^d} g_\beta(x, y + zL_N) = \frac{1}{(4\pi\beta)^{\frac{d}{2}}} \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\frac{|x-y-zL_N|^2}{4\beta}}. \tag{A.1.3}$$

$\diamond$

For both periodic and Dirichlet boundary conditions, the Poisson intensities (2.2.9) are then replaced by

$$\overline{q}_N^{(\mathrm{bc},\alpha)} = \sum_{k=1}^{\infty} q_{N,k}^{(\mathrm{bc})} \mathrm{e}^{\beta\alpha k}, \qquad \text{with } q_{N,k}^{(\mathrm{bc})} = \frac{1}{k|\Lambda_N|} \int_{\Lambda_N} \mathrm{d}x g_{k\beta}^{(\mathrm{bc},\mathrm{N})}(x,x). \tag{A.1.4}$$

A.2 Cycle Count Reference Process

When we consider the empirical cycle count, the weights $q_{N,k}^{(\mathrm{bc})}$ become very important. We can now consider the reference process to be a random process on ℓ_1, where each coordinate is an independent Poisson random variable with mean $|\Lambda_N| q_{N,k}^{(\mathrm{bc})}$. It will naturally be advantageous to have some more detail on the behaviour of these means. Recall that in both boundary condition cases $|\Lambda_N| = L_N^d$.

Lemma A.2.1. *For the Dirichlet boundary condition, we have the asymptotic sum*

$$q_{N,k}^{(\mathrm{Dir})} = \frac{1}{2^{\frac{d}{2}}|\Lambda_N|k} \exp\left(-\beta k \frac{\pi^2 d}{2L_N^2}\right) \left(\sum_{p=1}^{\infty} \exp\left(-\frac{\pi^2\beta k}{2L_N^2}\left(p^2-1\right)\right)\right)^d \tag{A.2.1}$$

$$\sim \frac{1}{2^{\frac{d}{2}}|\Lambda_N|k} \exp\left(-\beta k \frac{\pi^2 d}{2L_N^2}\right) \qquad \textit{for } k \gg L_N^2. \tag{A.2.2}$$

For the periodic boundary condition, we have the doubly infinite asymptotic sum

$$q_{N,k}^{(\mathrm{per})} = \frac{1}{|\Lambda_N|k} \left(\sum_{p=-\infty}^{\infty} \exp\left(-\frac{4\pi^2\beta k}{L_N^2} p^2\right)\right)^d \tag{A.2.3}$$

$$\sim \frac{1}{|\Lambda_N|k} \qquad \textit{for } k \gg L_N^2. \tag{A.2.4}$$

Proof. We begin with the Dirichlet case. First note that from (A.1.2) we have

$$g_{\beta k}^{(\mathrm{Dir}\,,N)}\left(x,x\right)=g_{\beta k}^{(\varnothing)}\left(x,x\right)\mathbb{P}_{x}\left(B_{[0,\beta k]}\subset\Lambda_{N}\right), \tag{A.2.5}$$

where $\mathbb{P}_{x}\left(B_{[0,\beta k]}\subset\Lambda_{N}\right)$ is the probability that a Brownian bridge (starting and ending at x with time horizon βk) stays in the set Λ_N. This means that the Poisson weights can be written as

$$q_{N,k}^{(\mathrm{Dir}\,)}=q_{k}\frac{1}{|\Lambda_{N}|}\int_{\Lambda_{N}}\mathrm{d}x\mathbb{P}_{x}\left(B_{[0,\beta k]}\subset\Lambda_{N}\right). \tag{A.2.6}$$

Recall that Λ_N is a d-cube with sides length L_N. Then the probability factorises into d independent probabilities relating to 1-dimensional Brownian bridges, and

$$\frac{1}{|\Lambda|}\int_{\Lambda}\mathrm{d}x\mathbb{P}_{x}\left(B_{[0,\beta k]}\subset\Lambda_{N}\right)=\left(\frac{2}{L_{N}}\int_{0}^{\frac{L_{N}}{2}}\mathrm{d}x\mathbb{P}_{x}\left(\max\left|B_{[0,\beta k]}\right|\leq\frac{L_{N}}{2}\right)\right)^{d}. \tag{A.2.7}$$

Now use the scaling properties of Gaussian processes to make the calculation clearer. In particular,

$$\frac{2}{L}\int_{0}^{\frac{L}{2}}\mathrm{d}x\mathbb{P}\left(\max\left|B_{t}^{\beta k,x}\right|\leq\frac{L}{2}\right)=\int_{0}^{1}\mathrm{d}xP\left(x,T\right), \tag{A.2.8}$$

where

$$P\left(x,T\right)=\mathbb{P}\left(\mathcal{B}_{[0,T]}\subset[-1-x,1-x]\right),\qquad T=\frac{4\beta k}{L_{N}^{2}}. \tag{A.2.9}$$

Here $\mathcal{B}_t$ is a standard Brownian bridge starting and ending at 0 with time horizon T. Specifically, we can construct it in terms of the standard Wiener process, $W\left(t\right)$, thus:

$$\mathcal{B}_{t}=\frac{\sqrt{T}}{1+r\left(t\right)}W\left(r\left(t\right)\right),\qquad r\left(t\right)=\frac{t}{T-t}. \tag{A.2.10}$$

See, for example, [SW86]. Note that r is a bijection between $[0,T)$ and $[0,\infty)$. Therefore

$$P\left(x,T\right)=\mathbb{P}\left(\frac{W\left(r\right)}{1+r}\in\left[\frac{-1-x}{\sqrt{T}},\frac{1-x}{\sqrt{T}}\right],\forall r\in[0,\infty)\right). \tag{A.2.11}$$

Probabilities of this form have been studied by Doob in [Doo49]. In particular, this source gives us the expression

$$P(x,T) = 1 - \sum_{m=1}^{\infty} \mathrm{e}^{-\frac{2}{T}(2m-1+x)^2} + \mathrm{e}^{-\frac{2}{T}(2m-1-x)^2} - 2\mathrm{e}^{-\frac{2}{T}(2m)^2} \tag{A.2.12}$$

$$= \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\frac{2}{T}(2m)^2} - \mathrm{e}^{-\frac{2}{T}(2m-1+x)^2}. \tag{A.2.13}$$

Note that all these terms converge absolutely.

We proceed by using the Poisson summation formula (for example in [DE14]). Define the *Schwartz space*, $\mathcal{S}(\mathbb{R})$, as the space of all C^∞ functions $f : \mathbb{R} \to \mathbb{C}$ such that for any two integers $m, n \geq 0$ the function $x^n f^{(m)}(x)$ is bounded. Then for all $f \in \mathcal{S}(\mathbb{R})$,

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{p=-\infty}^{\infty} \hat{f}(p), \tag{A.2.14}$$

where $\hat{f}(p) := \int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{-2\pi i y p} f(y)$ is the Fourier transform.

To apply this formula, let $f(m) = \mathrm{e}^{-\frac{2}{T}(2m-1+x)^2}$. Clearly $f \in \mathcal{S}(\mathbb{R})$. Then the Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} dy \mathrm{e}^{-\frac{2}{T}(2y-1+x)^2 - 2\pi i p y} = \mathrm{e}^{-(1-x)\pi i p} \int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{e}^{-\frac{8}{T}\tau^2 - 2\pi i p \tau}. \tag{A.2.15}$$

This last integral is the Fourier transform of a Gaussian. If we differentiate this integral with respect to p and use integration by parts, we find an ordinary differential equation with the unique solution

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{e}^{-\frac{8}{T}\tau^2 - 2\pi i p \tau} = \mathrm{e}^{-\frac{\pi^2 T}{8}p^2} \int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{e}^{-\frac{8}{T}\tau^2} = \sqrt{\frac{\pi T}{8}} \mathrm{e}^{-\frac{\pi^2 T}{8}p^2}. \tag{A.2.16}$$

We can then proceed with elementary calculations to find

$$P(x,T) = \sqrt{2\pi T} \sum_{p=1}^{\infty} \mathrm{e}^{-\frac{\pi^2 T}{8}p^2} \sin^2\left(\frac{p\pi}{2}(1-x)\right) \tag{A.2.17}$$

$$\int_0^1 \mathrm{d}x P(x,T) = \sqrt{\frac{\pi T}{2}} \sum_{p=1}^{\infty} \mathrm{e}^{-\frac{\pi^2 T}{8}p^2}. \tag{A.2.18}$$

This gives the asymptotic sum for the Dirichlet case as required, and the large k behaviour follows because

$$\sum_{p=2}^{\infty} \exp\left(-\frac{\pi^2\beta k}{2L_N^2}\left(p^2-1\right)\right) \to 0 \qquad \text{as } \frac{k}{L_N^2} \to \infty. \tag{A.2.19}$$

For the periodic case, we begin similarly. From the independence of the Cartesian coordinates of the Wiener process, we only need to consider the $d = 1$ case - higher dimensional cases will be powers of this. From (A.1.3) we have

$$g_{\beta k}^{(\mathrm{per},N)}(x,x) = \frac{1}{\sqrt{4\pi\beta k}} \sum_{m=-\infty}^{\infty} \exp\left(-\frac{m^2 L_N^2}{4\beta k}\right). \tag{A.2.20}$$

Now we once again use the Poisson summation formula (A.2.14), and

$$g_{\beta k}^{(\mathrm{per},N)}(x,x) = \frac{1}{L_N} \sum_{p=-\infty}^{\infty} \exp\left(-\frac{4\pi^2\beta k}{L_N^2} p^2\right). \tag{A.2.21}$$

Finally substituting this into (A.1.4) gives the desired expression. In the $k \gg L_N$ limit, only the $p = 0$ term remains.

□

Lemma A.2.2. *For* $\mathrm{bc} \in \{\mathrm{Dir}\,, \mathrm{per}\}$*, we have*

$$\left\{q_{N,k}^{(\mathrm{bc})}\right\}_{k\in\mathbb{N}} \xrightarrow{\ell_\infty} \left\{q_k\right\}_{k\in\mathbb{N}}. \tag{A.2.22}$$

If we also have $u < 0$*, then*

$$\left\{q_{N,k}^{(\mathrm{bc})}\mathrm{e}^{\beta uk}\right\}_{k\in\mathbb{N}} \xrightarrow{\ell_1} \left\{q_k\mathrm{e}^{\beta uk}\right\}_{k\in\mathbb{N}}. \tag{A.2.23}$$

PROOF. First note that from the expressions (A.1.2) and (A.1.3), it is clear that we have point-wise convergence. In the Dirichlet case we immediately have $0 < q_{N,k}^{(\mathrm{Dir})} \leq q_k$. Therefore by the Dominated Convergence Theorem we have ℓ_1-convergence even for $u = 0$. For the periodic case, we have from Lemma A.2.1 that $q_{N,k}^{(\mathrm{per})}$ is monotone decreasing in k (with vanishing limit) and from (A.1.3) we have that it is monotone decreasing in N. Now given $\epsilon > 0$, choose $K_\epsilon \in \mathbb{N}$ such that $q^{(\mathrm{per})}_{1,K_\epsilon} < \epsilon$. Then

$$\sup_{N\in\mathbb{N},k\geq K_\epsilon} q^{(\mathrm{per})}_{N,k} < \epsilon. \tag{A.2.24}$$

The point-wise convergence then deals with the $k < K_\epsilon$ terms and we have uniform convergence.

Now we get ℓ_1-convergence for $u < 0$ from the uniform convergence for $u = 0$. Given $\epsilon > 0$, choose N sufficiently large that $\sup_k \left\{q^{(\mathrm{bc})}_{N,k} - q_k\right\} < \epsilon\left(\mathrm{e}^{-\beta u} - 1\right)$. Then

$$\sum_{k\in\mathbb{N}} \left|q^{(\mathrm{bc})}_{N,k}\mathrm{e}^{\beta uk} - q_k\mathrm{e}^{\beta uk}\right| = \sum_{k\in\mathbb{N}} \mathrm{e}^{\beta uk}\left|q^{(\mathrm{bc})}_{N,k} - q_k\right| < \epsilon\left(\mathrm{e}^{-\beta u} - 1\right)\sum_{k\in\mathbb{N}} \mathrm{e}^{\beta uk} = \epsilon, \tag{A.2.25}$$

as required.

□

Lemma A.2.3. *Let r_N be a sequence in $\mathbb{N}$ such that $r_N \to \infty$. Then for* $\mathrm{bc} \in \{\mathrm{Dir}, \mathrm{per}\}$,

$$\lim_{N\to\infty} \frac{1}{r_N} \log q^{(\mathrm{bc})}_{N,r_N} = 0. \tag{A.2.26}$$

Proof. For the Dirichlet case we have $0 < q^{(\mathrm{Dir})}_{N,k} \leq q_k$. In the periodic case we have that $q^{(\mathrm{per})}_{N,k}$ is monotone decreasing in k (with vanishing limit) and that it is monotone decreasing in N. Therefore the have the required limit superior result.

From (A.1.3) we have that $q^{(\mathrm{per})}_{N,k} \geq q_k$. Therefore the fact that

$$\frac{1}{r_N}\log q_{r_N} = -\frac{1+\frac{d}{2}}{r_N}\log r_N + O\left(\frac{1}{r_N}\right) \to 0, \tag{A.2.27}$$

proves this for the periodic case.

For the Dirichlet case, note that Lemma A.2.1 implies that

$$\frac{1}{r_N}\log q^{(\mathrm{Dir})}_{N,r_N} \geq -\frac{\pi^2 d\beta}{2L_N^2} - \frac{1}{r_N}\log r_N - \frac{d}{r_N}\log L_N + O\left(\frac{1}{r_N}\right). \tag{A.2.28}$$

Therefore for $r_N \gg \log L_N$ we have our result. For the remainder, we note

$$\frac{1}{r_N}\log q_{N,k}^{(\mathrm{Dir})} = \frac{1}{r_N}\log q_k + \frac{1}{r_N}\log\frac{q_{N,k}^{(\mathrm{Dir})}}{q_k} \tag{A.2.29}$$

$$= \frac{1}{r_N}\log q_k + \frac{d}{r_N}\log\int_0^1 \mathrm{d}x P\left(x,T\right), \tag{A.2.30}$$

where we have used the notation from the proof of Lemma A.2.1. We know that $P\left(x,T\right)\uparrow 1$ for all $x\in[0,1)$ as $T\to 0$, and therefore our result hold if $r_N \ll L_N^2$.

$\square$

Appendix B

Level-3 LDP with Negative Potential

Here we present the proof of Lemma 3.3.1, repeated here.

Lemma B.0.1. *Let* $\tilde{\Phi}^{(-)}$ *be the* $R \to \infty$ *pointwise limit of* $\Phi^{(-,R,\infty,\infty)}$. *Then*

$$\liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \log \mathtt{E}_\alpha \left[\mathrm{e}^{-H^{(+)}_{\Lambda_L} - H^{(-)}_{\Lambda_L}} \right] \geq - \inf_{P \in \mathcal{P}_\theta} \left\{ I_\alpha \left(P \right) + \left\langle P, \Phi^{(+)} + \tilde{\Phi}^{(-)} \right\rangle \right\}. \tag{B.0.1}$$

To show this we first bound the Hamiltonian by a continuous function.

Lemma B.0.2. *Fix* $\Lambda = \Lambda_L$. *Then for* $R, M, S \in (3, +\infty)$ *and* $L \geq R + 1$,

$$H_\Lambda \left(\omega \right) \leq |\Lambda| \left\langle \mathfrak{R}_{\Lambda,\omega}, \Phi^{(R,M,S)} \right\rangle + 6^d M S N_{\Lambda_L \setminus \Lambda_{L-R-1}} \left(\omega \right). \tag{B.0.2}$$

Proof. We proceed similarly to the proof of Lemma 3.1.10. We find

$$|\Lambda| \left\langle \mathfrak{R}_{\Lambda,\omega}, \Phi^{(R,M,S)} \right\rangle \geq H_\Lambda \left(\omega \right) + \Psi^{(R,M,S)}_\Lambda \left(\omega \right), \tag{B.0.3}$$

where

$$\Psi_{\Lambda}^{(R,M,S)}(\omega) = \sum_{x,y\in\xi_{(\Lambda)}:(x,y)\notin\Lambda^2} T_{x,y}^{(R,M)}\left(\omega_{(\Lambda)}\right)\int_{\Lambda\cap(U-x)} \mathbb{1}\left\{\#\left(\xi_{(\Lambda)}\cap\left(\Lambda_R - z\right)\right)\leq S\right\} \tag{B.0.4}$$

$$\geq -M \sum_{\substack{x,y\in\xi_{(\Lambda)}:\\ x-y\in\Lambda_R,(x,y)\notin\Lambda^2}} \mathbb{1}\left\{\#\left(\xi_{(\Lambda)}\cap\left(\Lambda_{R-1} - x\right)\right)\leq S\right\} \tag{B.0.5}$$

$$\geq -6^d MSN_{\Lambda_L\setminus\Lambda_{L-R-1}}(\omega). \tag{B.0.6}$$

□

By a Hölder-type inequality, Lemma B.0.2 gives us

$$\mathtt{E}_\alpha\left[\mathrm{e}^{-H_\Lambda}\right] \geq \mathtt{E}_\alpha\left[\exp\left(-\left(1-\eta\right)|\Lambda|\left\langle\mathfrak{R}_{\Lambda,\cdot},\Phi^{(R,M,S)}\right\rangle\right)\right]^{\frac{1}{1-\eta}} \times \mathtt{E}_\alpha\left[\exp\left(\frac{1-\eta}{\eta}6^d MSN_{\Lambda_L\setminus\Lambda_{L-R-1}}\right)\right]^{\frac{-\eta}{1-\eta}}, \tag{B.0.7}$$

for $\eta\in(0,1)$. Now, under the reference measure, $N_{\Lambda_L\setminus\Lambda_{L-R-1}}$ is a Poisson random variable with mean $\overline{q}|\Lambda_L\setminus\Lambda_{L-R-1}| = O\left(L^{d-1}\right)$. Therefore

$$1 \geq \mathtt{E}_\alpha\left[\exp\left(\frac{1-\eta}{\eta}6^d MSN_{\Lambda_L\setminus\Lambda_{L-R-1}}\right)\right]^{\frac{-\eta}{1-\eta}} = \exp\left(-\frac{\eta}{1-\eta}\overline{q}|\Lambda_L\setminus\Lambda_{L-R-1}|\left(\mathrm{e}^{\frac{1-\eta}{\eta}6^d MS}-1\right)\right) \geq \mathrm{e}^{-o(|\Lambda|)}, \tag{B.0.8}$$

so we can neglect this factor.

Let us define $F:\mathcal{P}\to[-\infty,0]$, $F_{R,M,S,\eta}:\mathcal{P}\to[-\infty,0]$, and $F_{\mathrm{loc}}:\mathcal{P}\to[-\infty,0]$ by

$$F: P\mapsto\left\langle P,\Phi^{(-)}\right\rangle \tag{B.0.9}$$

$$F_{R,M,S,\eta}: P\mapsto\frac{1}{1-\eta}\left\langle P,\Phi^{(-,R,M,S)}\right\rangle \tag{B.0.10}$$

$$F_{\mathrm{loc}}: P\mapsto\left\langle P,\tilde{\Phi}^{(-)}\right\rangle, \tag{B.0.11}$$

Since $\Phi^{(-,R,M,S)}$ is local and tame ($|\Phi^{(-,R,M,S)}|\leq MSN_{\Lambda_{R+1}}$), $F_{R,M,S,\eta}$ is $\tau_{\mathcal{L}}$-continuous.

We can apply Varadhan's Lemma to get

$$\liminf_{L\to\infty}\frac{1}{|\Lambda_L|}\log \mathtt{E}_\alpha\left[\exp\left(-|\Lambda_L|F_{R,M,S,\eta}\left(\mathfrak{R}_{\Lambda,\cdot}\right)\right)\right]\geq -\inf_{P\in\mathcal{P}_\theta}\left\{I\left(P\right)+F_{R,M,S,\eta}\left(P\right)\right\}. \tag{B.0.12}$$

It now suffices to prove the following:

Lemma B.0.3.

$$\liminf_{R\to\infty}\liminf_{M,S\to\infty,\eta\downarrow 0}\inf_{P\in\mathcal{P}_\theta}\left\{I\left(P\right)+F_{R,M,S,\eta}\left(P\right)\right\}\leq\inf_{P\in\mathcal{P}_\theta}\left\{I\left(P\right)+F_{\mathrm{loc}}\left(P\right)\right\}. \tag{B.0.13}$$

Proof. First let choose a sequence Q_n such that

$$I\left(Q_n\right)+F_{R,\infty,\infty,0}\left(Q_n\right)<\inf_{P\in\mathcal{P}_\theta}\left\{I\left(P\right)+F_{R,\infty,\infty,0}\left(P\right)\right\}+\frac{1}{n}. \tag{B.0.14}$$

Now fix R. Then $F_{R,M,S,\eta}$ is decreasing in M and S, and increasing in η, with pointwise convergence to $F_{R,\infty,\infty,0}$. Therefore there exist increasing sequences M_n and S_n, and a decreasing sequence η_n such that

$$I\left(Q_n\right)+F_{R,M_n,S_n,\eta_n}\left(Q_n\right)<I\left(Q_n\right)+F_{R,\infty,\infty,0}\left(Q_n\right)+\frac{1}{n}. \tag{B.0.15}$$

This means that we have found sequences M_n, S_n, η_n such that

$$\inf_{P\in\mathcal{P}_\theta}\left\{I\left(P\right)+F_{R,M_n,S_n,\eta_n}\left(P\right)\right\}<\inf_{P\in\mathcal{P}_\theta}\left\{I\left(P\right)+F_{R,\infty,\infty,0}\left(P\right)\right\}+\frac{2}{n}. \tag{B.0.16}$$

Because $\tilde{\Phi}^{(-)}$ is the pointwise limit of $\Phi^{(-,R,\infty,\infty)}$, we can apply Fatou's Lemma to get

$$\limsup_{R\to\infty}F_{R,\infty,\infty,0}\left(P\right)\leq F_{\mathrm{loc}}\left(P\right). \tag{B.0.17}$$

Since $F_{R,\infty,\infty,0}$ is decreasing in R, it converges pointwise to F_{loc} and we can repeat the above argument similarly.

□

Appendix C

Misc. Functions

C.1 Bose function

The Bose functions are poly-logarithmic functions defined by

$$g(n,\alpha) := \mathrm{Li}_{\,n}(\mathrm{e}^{-\alpha}) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{t^{n-1}}{\mathrm{e}^{t+\alpha}-1}\,\mathrm{d}t = \sum_{k=1}^{\infty} k^{-n}\mathrm{e}^{-\alpha k}, \qquad \forall n\in\mathbb{Z}, \alpha>0, \tag{C.1.1}$$

and also for $\alpha = 0$ and $n > 1$. In the latter case,

$$g(n,0) = \sum_{k=1}^{\infty} k^{-n} = \zeta(n), \tag{C.1.2}$$

which is the zeta function of Riemann. The behaviour of the Bose functions about $\alpha = 0$ is given by

$$g(n,\alpha) = \begin{cases} \Gamma(1-n)\alpha^{n-1} + \sum_{k=0}^{\infty} \zeta(n-k)\frac{(-\alpha)^k}{k!} & : n \neq 1,2,3,\ldots, \\ \frac{(-\alpha)^{n-1}}{(n-1)!}\left[-\log\alpha + \sum_{m=1}^{n-1}\frac{1}{m}\right] + \sum_{\substack{k=0\\ k\neq n-1}} \zeta(n-k)\frac{(-\alpha)^k}{k!} & : n\in\mathbb{N}. \end{cases} \tag{C.1.3}$$

At $\alpha = 0$, $g(n,\alpha)$ diverges for $n \leq 1$, indeed for all n there is some kind of singularity at $\alpha = 0$, such as a branch point. For further details see [GN25]. The expansions (C.1.3) are in terms of $\zeta(n)$, which for for $n \leq 1$ must be found by analytic ally continuing (C.1.2). With the asymptotic properties of the zeta function it can be shown that the k series in (C.1.3) are convergent for $|\alpha| < 2\pi$. Consequently (C.1.3) also represents an analytic continuation of $g(n,\alpha)$ for $\alpha < 0$. When $\alpha \gg 1$ the series (C.1.1) itself is rapidly convergent, and as $\alpha \to \infty$, $g(n,\alpha) \sim \mathrm{e}^{-\alpha}$ for all n. Some plots for the Bose functions are given in Figure C.1.

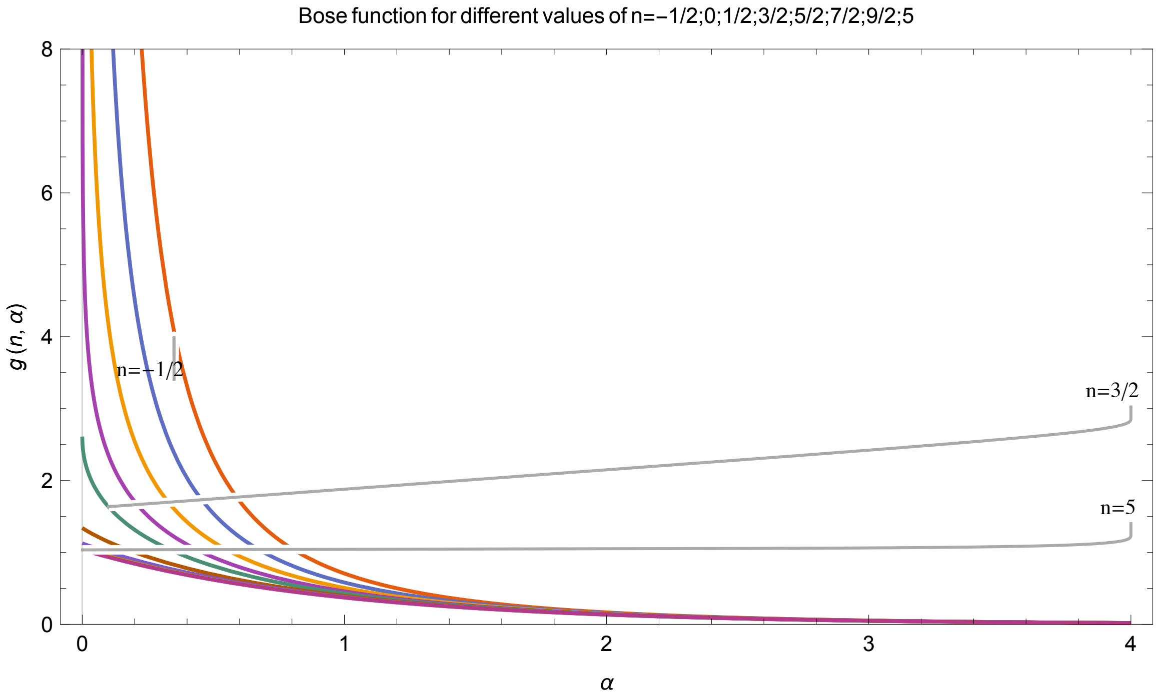


Figure C.1: From [AD18].

C.2 Lambert W function

The Lambert W function (sometimes called elsewhere the *Omega function*) is defined as the multi-valued inverse of the $\mathbb{C} \to \mathbb{C}$ function $w \mapsto w\mathrm{e}^{w}$. We shall only be concerned with the two branches on $\mathbb{R}$. Figure C.2 shows these two real branches, denoted W_0 and W_{-1}. The W_0 branch is defined on $[-\mathrm{e}^{-1}, \infty)$, whereas the W_{-1} branch is only defined on $[-\mathrm{e}^{-1}, 0)$. Given a branch W_l with $l \in \{0, -1\}$, we can find its (real) derivative W_l' by differentiating the equation $W_l(x)\,\mathrm{e}^{W_l(x)} = x$. This gives us

$$W_l'(x) = \frac{1}{x}\frac{W_l(x)}{1 + W_l(x)}. \tag{C.2.1}$$

Taking further derivatives and applying induction shows that the branches are smooth on the interior of their respective domains, and gives expressions for each

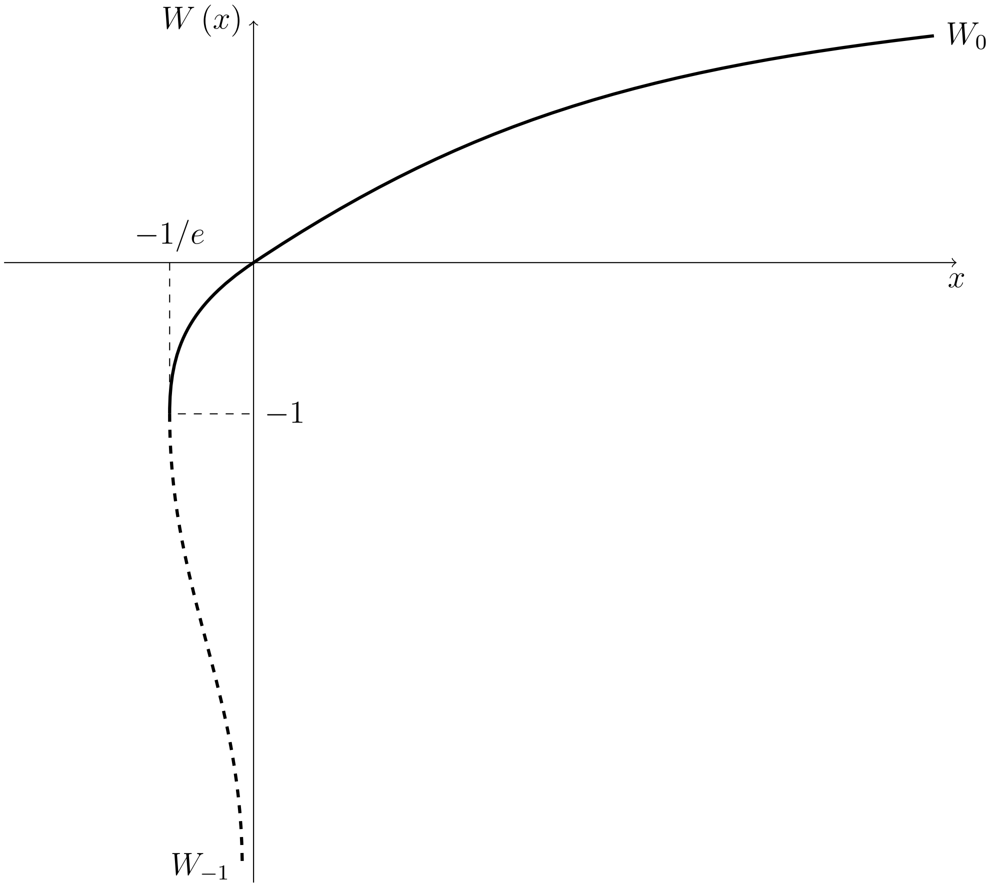


Figure C.2: The two real branches of W: W_0 and W_{-1}.

order of the derivative. We make use of some asymptotic expansions of W_0 and W_1:

$$
\begin{aligned}
W_0(x) &= x - x^2 + o\left(x^2\right) && \text{as } x \to 0,\\
W_0(x) &= \log x - \log(\log x) + o(1) && \text{as } x \to +\infty, \qquad \text{(C.2.2)}\\
W_{-1}(x) &= \log(-x) - \log(-\log(-x)) + o(1) && \text{as } x \uparrow 0.
\end{aligned}
$$

For more details, see [Cor+96].

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